Lecture 27: May 17

Today is the last class of the semester. We are going to finish the proof of Theorem 26.4. Let me state the result again.

Theorem. *Let $f\colon X \to A$ be a morphism from a smooth projective variety to an abelian variety. If $H^0(X, \omega_X^m \otimes f^*L^{-1}) \neq 0$ for some $m \geq 1$ and some ample line bundle L on A, then one has $Z(f^*\omega) \neq \emptyset$ for every $\omega \in H^0(A, \Omega_A^1)$.*

Last time, we introduced the set

$$S_f = \big\{ (a, \omega) \in A \times W \bigm| f^{-1}(a) \cap Z(f^*\omega) \neq \emptyset \big\} = (f \times \mathrm{id})\big(df)^{-1}(0)\big),$$

where the notation is as follows:

$$\begin{array}{ccc} X \times W & \xrightarrow{df} & T^*X \\ \big\downarrow{\scriptstyle f \times \mathrm{id}} & & \\ A \times W & & \end{array}$$

We also observed that the result about one-forms is equivalent to the surjectivity of $p_2\colon S_f \to W$. Finally, we talked briefly about filtered $\mathscr{D}$-modules and Hodge modules, and we showed that if $(\mathcal{M}, F_\bullet\mathcal{M})$ is a Hodge module on the abelian variety A, and if $\mathscr{G}$ is the coherent sheaf on $T^*A = A \times W$ corresponding to $\mathrm{gr}^F\mathcal{M}$, then for any ample line bundle L,

$$(p_2)_*(\mathscr{G} \otimes p_1^*L^{-1})$$

is a *torsion-free* coherent sheaf on W. This was a consequence of Saito's vanishing theorem, ultimately. Today, I will show you how to construct the required objects from the hypothesis that $\omega_X^m \otimes f^*L^{-1}$ has a section.

Base change. Whenever the m-th power of a line bundle has a section, one can construct a cyclic covering. We can put ourselves in this situation with the help of a very useful small trick. On the abelian variety A, we have the multiplication homomorphism

$$[m]\colon A \to A, \quad a \mapsto \underbrace{a + \cdots + a}_{m \text{ times}},$$

for any $m \in \mathbb{Z}$. It is finite and étale, of degree equal to $m^{2\dim A}$, which is the same as the number of m-torsion points in A. The effect of pulling back by $[m]$ is to make line bundles more divisible. In fact, if L is symmetric, in the sense that $[-1]^*L \cong L$, then one has $[m]^*L \cong L^{m^2}$; if L is anti-symmmetric, in the sense that $[-1]^*L \cong L^{-1}$, then one still has $[m]^*L \cong L^m$. Since we can write any line bundle as the product of a symmetric and an anti-symmetric one, it follows that

$$[2m]^*L \cong L'^m$$

for some other line bundle L'. Now consider the fiber product diagram

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow{[2m]} & A. \end{array}$$

Because ψ is finite and étale, we get $\psi^*\omega_X \cong \omega_{X'}$, and therefore

$$\psi^*\big(\omega_X^m \otimes f^*L^{-1}\big) \cong \big(\omega_{X'} \otimes f'^*L'^{-1}\big)^m.$$

Again because ψ is finite and étale, it does not affect the zero loci of holomorphic one-forms; more precisely, we have

$$\psi^{-1}Z(f^*\omega) = Z(f'^*\omega),$$

because $[2m]^*\omega = 2m\cdot\omega$. For the purpose of proving Theorem 26.4, we can therefore safely replace $f\colon X \to A$ by its base change $f'\colon X' \to A$; this allows us to assume that the m-th power of the line bundle $B = \omega_X \otimes f^*L^{-1}$ has a nontrivial global section.

Cyclic coverings. Suppose for a moment that we have a nonsingular algebraic variety X and a line bundle B, as well as a nontrivial global section $s \in H^0(X, B^m)$ for some $m \geq 2$. In that case, one can construct a finite morphism

$$\pi\colon Y \to X$$

with the property that π^*B has a global section s_0 such that $s_0^m = \pi s$. Since the group of m-th roots of unities naturally acts on Y, this is called the *cyclic covering* determined by the section s.

Example 27.1. When B is the trivial bundle, s is just a regular function on X; in that case, Y is the closed subscheme of $X \times \mathbb{A}^1$ defined by the equation $t^m = s$, where t is the coordinate on $\mathbb{A}^1$. Here t serves as the m-th root of s.

The construction in the general case is similar. Let $p\colon V = \mathbb{V}(B) \to X$ be the algebraic line bundle (whose sheaf of sections is the locally free sheaf B). The pullback π^*B has a tautological section $s_0 \in H^0(V, \pi^*B)$, and one defines $Y \subseteq V$ as the closed subscheme cut out by the section $s_0^m - \pi^*s$ of the line bundle π^*B^m. By construction, the morphism $\pi\colon Y \to X$ is finite of degree m, and π^*B has a global section s_0 such that $s_0^m = \pi^*s$. (This construction has a simple universal property, which I will leave to you to formulate and prove.)

Unless the divisor of s is nonsingular, the cyclic covering Y will be singular, but we can resolve its singularities. In this way, we obtain a proper morphism

$$\varphi\colon Z \to X,$$

generically finite of degree m, from a nonsingular algebraic variety Z, such that the line bundle φ^*B has a section $s_0 \in H^0(Z, \varphi^*B)$ with $s_0^m = \varphi^*s$.

Sheaves. If we apply the cyclic covering construction to $B = \omega_X \otimes f^*L^{-1}$, we obtain the following diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\ \varphi\ } & X \\ & \searrow^{h} & \downarrow{\scriptstyle f} \\ & & A \end{array}$$

Here Z is a nonsingular projective variety of dimension $\dim Z = \dim X = n$, and φ is generically finite of degree m. We may view the resulting nontrivial section of $\varphi^*B = \varphi^*\omega_X \otimes h^*L^{-1}$ as a nontrivial morphism

$$h^*L \to \varphi^*\omega_X. \tag{27.2}$$

We can use the morphism from Z to A to construct a filtered $\mathscr{D}$-module on the abelian variety. The underlying $\mathscr{D}_A$-module is simply the direct image $\mathcal{M} = \mathcal{H}^0 h_+ \omega_Z$. Since $(\omega_Z, F_\bullet\omega_Z)$ is actually a Hodge module on Z, the graded $\tilde{\mathscr{D}}_A$-module $\tilde{\mathcal{M}} = \mathcal{H}^0 h_+(R_F\omega_Z)$ is strict, and so there is a good filtration $F_\bullet\mathcal{M}$ such that $\tilde{\mathcal{M}} = R_F\mathcal{M}$. Moreover, $(\mathcal{M}, F_\bullet\mathcal{M})$ is again a Hodge module on A. If we denote by $\mathscr{G}$ the associated coherent sheaf on $T^* = A \times W$, then we know from last time that

$$(p_2)_*\big(\mathscr{G} \otimes p_1^*L^{-1}\big)$$

is a torsion-free coherent sheaf on W.

Since we constructed $\mathscr{G}$ from the morphism $h\colon Z \to A$, which is more singular than the original morphism $f\colon X \to A$, the support of $\mathscr{G}$ has nothing to do with the set $S_f \subseteq T^*A$ that we are interested in; in fact, one has $\operatorname{Supp}\mathscr{G} \subseteq S_h$, which is much larger in general. But we can use the existence of (27.2) to construct another coherent sheaf $\mathscr{F}$ with $\operatorname{Supp}\mathscr{F} \subseteq S_f$. Consider again the "big" diagram

$$\begin{array}{ccccc} Z \times W & \longrightarrow & Z \times_X T^*X & \xrightarrow{d\varphi} & T^*Z \\ \big\downarrow{\scriptstyle \varphi\times\mathrm{id}} & & \big\downarrow{\scriptstyle p_2} & & \\ X \times W & \xrightarrow{df} & T^*X & & \\ \big\downarrow{\scriptstyle f\times\mathrm{id}} & & & & \\ A \times W. & & & & \end{array}$$

$$dh\colon Z\times W \to T^*Z, \qquad h\times\mathrm{id}\colon Z\times W \to A\times W$$

Last time, we said that for direct images of Hodge modules, one can compute the corresponding sheaves on the cotangent bundle very explicitly. The characteristic variety of ω_Z is the zero section in T^*Z, and the resulting coherent sheaf is $i_*\omega_Z$, where $i\colon Z \hookrightarrow T^*Z$ is the zero section. In the case of $\mathcal{M} = \mathcal{H}^0 h_+\omega_Z$, the formula from last time says that $\mathscr{G}$ is the 0-th cohomology sheaf of the complex

$$\mathbf{R}(h\times \mathrm{id})_*\mathbf{L}(dh)^*(i_*\omega_Z).$$

Let $p\colon T^*Z \to Z$ be the projection. Since the zero section is exactly the vanishing locus of the tautological section of $p^*\Omega_Z^1$, the Koszul complex

$$p^*\Omega_Z^{n+\bullet} = \Big[p^*\mathscr{O}_Z \to p^*\Omega_Z^1 \to \dots \to p^*\Omega_Z^n\Big]$$

is a locally free resolution of the coherent sheaf $i_*\omega_Z$ on T^*Z. Consequently,

$$\mathbf{L}(dh)^*(i_*\omega_Z) = \Big[p_1^*\mathscr{O}_Z \to p_1^*\Omega_Z^1 \to \dots \to p_1^*\Omega_Z^n\Big],$$

which means that $\mathscr{G}$ is the 0-th cohomology sheaf of the complex

$$\mathbf{R}(h\times\mathrm{id})_*\Big[p_1^*\mathscr{O}_Z \to p_1^*\Omega_Z^1 \to \dots \to p_1^*\Omega_Z^n\Big].$$

Now consider the morphism $\varphi\colon Z \to X$. For each $p \geq 0$, we have a pullback morphism $\varphi^*\Omega_X^p \to \Omega_Z^p$; these fit together into a morphism of complexes

$$\Big[p_1^*\varphi^*\mathscr{O}_X \to p_1^*\varphi^*\Omega_X^1 \to \dots \to p_1^*\varphi^*\Omega_X^n\Big] \to \Big[p_1^*\mathscr{O}_Z \to p_1^*\Omega_Z^1 \to \dots \to p_1^*\Omega_Z^n\Big].$$

In derived category notation, this means that we have a morphism

$$\mathbf{L}(\varphi\times\mathrm{id})^*\mathbf{L}(df)^*(i_*\omega_X) \to \mathbf{L}(dh)^*(i_*\omega_Z).$$

Here $i\colon X \hookrightarrow T^*X$ is the zero section, and $p\colon T^*X \to X$ the projection. Since $i_*\mathscr{O}_X \otimes p^*\omega_X \cong i_*(\mathscr{O}_X \otimes i^*p^*\omega_X) \cong i_*\omega_X$ by the projection formula, we can rewrite this morphism in the more convenient form

$$p_1^*(\varphi^*\omega_X) \otimes \mathbf{L}(\varphi\times\mathrm{id})^*\mathbf{L}(df)^*(i_*\mathscr{O}_X) \to \mathbf{L}(dh)^*(i_*\omega_Z).$$

Now we compose this with (27.2) to obtain a morphism

$$p_1^*(h^*L) \otimes \mathbf{L}(\varphi\times\mathrm{id})^*\mathbf{L}(df)^*(i_*\mathscr{O}_X) \to \mathbf{L}(dh)^*(i_*\omega_Z).$$

Move the line bundle factor to the other side, and use the adjunction between the two functors $\mathbf{L}(\varphi\times\mathrm{id})^*$ and $\mathbf{R}(\varphi\times\mathrm{id})_*$. This gives an equivalent morphism

$$\mathbf{L}(df)^*(i_*\mathscr{O}_X) \to \mathbf{R}(\varphi\times\mathrm{id})_*\Big(p_1^*(h^*L^{-1}) \otimes \mathbf{L}(dh)^*(i_*\omega_Z)\Big).$$

Now push forward to $A \times W$ and use the projection formula to pull out the line bundle factor. This finally gives us the following morphism

$$\mathbf{R}(f\times\mathrm{id})_*\mathbf{L}(df)^*(i_*\mathscr{O}_X) \to \mathbf{R}(h\times\mathrm{id})_*\mathbf{L}(dh)^*(i_*\omega_Z) \otimes p_1^*L^{-1} \tag{27.3}$$

in the derived category $D^b_{coh}(\mathscr{O}_{A\times W})$. If we take cohomology in degree zero, we therefore obtain a morphism of coherent sheaves

$$\mathscr{F} \to \mathscr{G} \otimes p_1^* L^{-1}. \tag{27.4}$$

Here $\mathscr{F}$ is the 0-th cohomology sheaf of the complex $\mathbf{R}(f\times \mathrm{id})_* \mathbf{L}(df)^*(i_*\mathscr{O}_X)$, and as such, it is obviously supported inside the set

$$(f\times \mathrm{id})\bigl(df^{-1}(0)\bigr) = S_f.$$

Now all the pieces are in place to prove the theorem about one-forms.

Proof of Theorem 26.4. We are trying to show that $p_2\colon S_f \to W$ is surjective. Suppose, for the sake of argument, that $p_2(S_f) \neq W$. Then $(p_2)_*\mathscr{F}$ is a coherent sheaf on W whose support is contained inside a proper closed subset, hence a torsion sheaf. Because $(p_2)_*(\mathscr{G}\otimes p_1^*L^{-1})$ is torsion-free, the morphism

$$(p_2)_*\mathscr{F} \to (p_2)_*(\mathscr{G}\otimes p_1^*L^{-1})$$

must be trivial. Taking global sections, this means that the morphism

$$H^0(A\times W, \mathscr{F}) \to H^0(A\times W, \mathscr{G}\otimes p_1^*L^{-1})$$

is also trivial. Now both sides are actually graded modules, due to the fact that (27.3) is constructed from sheaves on the zero section (which are stable under the natural $\mathbb{C}^*$-action on the cotangent bundle). The first nontrivial graded piece (in degree $-n$) comes out to be

$$H^0(X, \mathscr{O}_X) \to H^0(Z, \omega_Z \otimes h^*L^{-1})$$

But now we have a contradiction, because the composition $h^*L \to \varphi^*\omega_X \to \omega_Z$ is not the zero morphism, due to the fact that (27.2) is nontrivial by assumption. This means that we have a nontrivial section of $\omega_Z\otimes h^*L^{-1}$, and so the above morphism cannot have been zero. The conclusion is that $p_2(S_f) = W$. □