Lecture 26 (May 1)

Let me quickly remind you of the construction of Weil classes from last time. Let E be a CM-field, and $S = \operatorname{Hom}(E, \mathbb{C})$ the set of its complex embeddings. We write $e \mapsto \bar{e}$ for the involution on E; for any $s \in S$, we then have $\overline{s(e)} = s(\bar{e})$.

Let V be a rational Hodge structure of type $\{(1,0),(0,1)\}$ whose endomorphism algebra contains E. We shall assume that $\dim_E V = d$ is an even number; then $\dim_{\mathbb{Q}} V = d \cdot [E\colon \mathbb{Q}]$. For every $s \in S$, we define $V_s = V \otimes_{E,s} \mathbb{C}$, which is a complex vector space of dimension d. The tensor product gives us the relation $ev \otimes z = v \otimes s(e)z$. Corresponding to the decomposition

$$E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} \mathbb{C}, \quad e \otimes z \mapsto \sum_{s \in S} s(e)z,$$

we get a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} V_s.$$

Complex conjugation, which acts on the left-hand side as $v \otimes z \mapsto v \otimes \bar{z}$, exchanges the summand V_s with the summand $V_{\bar{s}}$; this can be seen by conjugating the identity $ev \otimes z = v \otimes s(e)z$ in V_s. By assumption, E respects the Hodge decomposition on V, and so we get an induced decomposition

$$V_s = V_s^{1,0} \oplus V_s^{0,1}.$$

Note that $\dim_{\mathbb{C}} V_s^{1,0} + \dim_{\mathbb{C}} V_s^{0,1} = d$.

Lemma 26.1. *The rational subspace $\bigwedge_E^d V \subseteq \bigwedge_{\mathbb{Q}}^d V$ is purely of type $(d/2, d/2)$ if and only if $\dim_{\mathbb{C}} V_s^{1,0} = \dim_{\mathbb{C}} V_s^{0,1} = d/2$ for every $s \in S$.*

When the condition in the lemma is satisfied, the subspace $\bigwedge_E^d V$ consists entirely of Hodge classes. These Hodge classes are called *Weil classes*. We are now going to give a linear algebra condition for this to be the case, using hermitian forms and polarizations.

Hermitian forms. This requires a little bit of background on hermitian forms. Throughout, E is a CM-field, with totally real subfield F and complex conjugation $e \mapsto \bar{e}$, and $S = \operatorname{Hom}(E, \mathbb{C})$ is the set of complex embeddings of E. An element $\zeta \in E^\times$ is called *totally imaginary* if $\bar{\zeta} = -\zeta$; concretely, this means that $\bar{s}(\zeta) = -s(\zeta)$ for every complex embedding s.

Definition 26.2. Let V be an E-vector space. A $\mathbb{Q}$-bilinear form $\phi\colon V \times V \to E$ is said to be *E-hermitian* if $\phi(e \cdot v, w) = e \cdot \phi(v, w)$ and $\phi(v, w) = \overline{\phi(w, v)}$ for every $v, w \in V$ and every $e \in E$. It follows that $\phi(v, e \cdot w) = \bar{e} \cdot \phi(v, w)$.

Now suppose that V is an E-vector space of dimension $d = \dim_E V$, and that ϕ is an E-hermitian form on V. We begin by describing the numerical invariants of the pair (V, ϕ). For any embedding $s\colon E \hookrightarrow \mathbb{C}$, we obtain a hermitian form ϕ_s in the usual sense on the complex vector space $V_s = V \otimes_{E,s} \mathbb{C}$. Concretely, we have

$$\phi_s\left(\sum_j v_j \otimes z_j, \sum_k v_k' \otimes z_k' \right) = \sum_{j,k} z_j \bar{z}_k' s\bigl(\phi(v_j, v_k')\bigr).$$

We let a_s and b_s be the dimensions of the maximal subspaces where ϕ_s is, respectively, positive and negative definite. Because $\dim_{\mathbb{C}} V_s = d$, the signature of the hermitian form ϕ_s is then $(a_s, b_s, d - a_s - b_s)$.

A second invariant of ϕ is its discriminant. To define it, note that ϕ induces an E-hermitian form on the one-dimensional E-vector space $\bigwedge_E^d V$, which up to a choice of basis vector, is of the form $(x, y) \mapsto f x \bar{y}$. The element f belongs to the totally real subfield F, and a different choice of basis vector only changes f by elements of the form $\mathrm{Nm}_{E/F}(e) = e \cdot \bar{e}$. Consequently, the class of f in $F^\times / \mathrm{Nm}_{E/F}(E^\times)$ is well-defined, and is called the *discriminant* of (V, ϕ). We denote it by the symbol $\operatorname{disc} \phi$. Equivalently, we can choose a basis for V and represent ϕ by a $d \times d$-matrix with entries in E; then $\operatorname{disc} \phi$ is the determinant of this matrix.

Now suppose that ϕ is nondegenerate. Let $v_1, \dots, v_d \in V$ be an orthogonal basis for ϕ, and set $c_i = \phi(v_i, v_i)$. Then we have $c_i \in F^\times$, and

$$a_s = \#\{\, i \mid s(c_i) > 0 \,\} \quad \text{and} \quad b_s = \#\{\, i \mid s(c_i) < 0 \,\}$$

satisfy $a_s + b_s = d$. Moreover, we have

$$f = \prod_{i=1}^{d} c_i \mod \mathrm{Nm}_{E/F}(E^\times);$$

this implies that $\operatorname{sgn}\bigl(s(f)\bigr) = (-1)^{b_s}$ for every $s \in S$. The following theorem by Landherr says that the discriminant and the integers a_s and b_s are a complete set of invariants for E-hermitian forms.

Theorem 26.3 (Landherr)**.** *Let $a_s, b_s \geq 0$ be a collection of integers, indexed by the set S, and let $f \in F^\times / \mathrm{Nm}_{E/F}(E^\times)$ be an arbitrary element. Suppose that they satisfy $a_s + b_s = d$ and $\operatorname{sgn}\bigl(s(f)\bigr) = (-1)^{b_s}$ for every $s \in S$. Then there exists a nondegenerate E-hermitian form ϕ on an E-vector space V of dimension d with these invariants; moreover, (V, ϕ) is unique up to isomorphism.*

This classical result has the following useful consequence.

Corollary 26.4. *If (V, ϕ) is nondegenerate, then the following two conditions are equivalent:*

- *(a) $a_s = b_s = d/2$ for every $s \in S$, and $\operatorname{disc} \phi = (-1)^{d/2}$.*
- *(b) There is a totally isotropic subspace of V of dimension $d/2$.*

Proof. If $W \subseteq V$ is a totally isotropic subspace of dimension $d/2$, then $v \mapsto \phi(-, v)$ induces an antilinear isomorphism $V/W \xrightarrow{\sim} W^\vee$. Thus we can extend a basis $v_1, \dots, v_{d/2}$ of W to a basis $v_1, \dots, v_d$ of V, with the property that

$$\begin{aligned} \phi(v_i, v_{i+d/2}) &= 1 \qquad \text{for } 1 \leq i \leq d/2, \\ \phi(v_i, v_j) &= 0 \qquad \text{for } |i - j| \neq d/2. \end{aligned}$$

We can use this basis to check that (a) is satisfied. For the converse, consider the hermitian space $(E^{\oplus d}, \phi)$, where

$$\phi(x, y) = \sum_{1 \leq i \leq d/2} \bigl(x_i \bar{y}_{i+d/2} + x_{i+d/2} \bar{y}_i \bigr)$$

for every $x, y \in E^{\oplus d}$. By Landherr's theorem, this space is (up to isomorphism) the unique hermitian space satisfying (a), and it is easy to see that it satisfies (b), too. ∎

Definition 26.5. An E-hermitian form ϕ that satisfies the two equivalent conditions in Corollary 26.4 is said to be *split*.

We shall see below that E-hermitian forms are related to polarizations on Hodge structures of CM-type. We now describe one additional technical result that is going tobe useful in that context. Suppose that V is a Hodge structure of type $\{(1,0),(0,1)\}$ that is of CM-type and whose endomorphism ring contains E; let $h \colon U(1) \to E^\times$ be the corresponding homomorphism. Recall that a *Riemann form* for V is a $\mathbb{Q}$-bilinear antisymmetric form $\psi \colon V \otimes V \to \mathbb{Q}$, with the property that

$$(x, y) \mapsto \psi\bigl(h(i) \cdot x, \bar{y}\bigr)$$

is hermitian and positive definite on $V \otimes_{\mathbb{Q}} \mathbb{C}$. We only consider Riemann forms whose Rosati involution induces complex conjugation on E, meaning that

$$\psi(ev, w) = \psi(v, \bar{e}w).$$

The next result says that polarizations with that property are closely related to E-hermitian forms.

Lemma 26.6. *Let $\zeta \in E^\times$ be a totally imaginary element ($\bar{\zeta} = -\zeta$), and let ψ be a Riemann form for V as above. Then there exists a unique E-hermitian form ϕ with the property that $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta\phi)$.*

Because the trace can be computed by summing over all complex embeddings, the formula $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ means concretely that

$$\psi(v, w) = \sum_{s \in S} s(\zeta) s\bigl(\phi(v, w)\bigr).$$

I did not give the proof in class, but I will include it here. We first prove a simpler statement about bilinear forms.

Lemma 26.7. *Let V and W be finite-dimensional vector spaces over E, and let $\psi \colon V \times W \to \mathbb{Q}$ be a $\mathbb{Q}$-bilinear form such that $\psi(ev, w) = \psi(v, ew)$ for every $e \in E$. Then there exists a unique E-bilinear form ϕ such that $\psi(v, w) = \operatorname{Tr}_{E/\mathbb{Q}} \phi(v, w)$.*

Proof. The trace pairing $E \times E \to \mathbb{Q}$, $(x, y) \mapsto \operatorname{Tr}_{E/\mathbb{Q}}(xy)$, is nondegenerate. Consequently, composition with $\operatorname{Tr}_{E/\mathbb{Q}}$ induces an injective homomorphism

$$\operatorname{Hom}_E\bigl(V \otimes_E W, E\bigr) \to \operatorname{Hom}_{\mathbb{Q}}\bigl(V \otimes_E W, \mathbb{Q}\bigr),$$

which has to be an isomorphism because both vector spaces have the same dimension over $\mathbb{Q}$. By assumption, ψ defines a $\mathbb{Q}$-linear map $V \otimes_E W \to \mathbb{Q}$, and we let ϕ be the element of $\operatorname{Hom}_E\bigl(V \otimes_E W, E\bigr)$ corresponding to ψ under the above isomorphism. $\square$

Proof of Lemma 26.6. We apply the preceding lemma with $W = V$, but with E acting on W through complex conjugation. This gives a sesquilinear form ϕ_1 such that $\psi(x, y) = \operatorname{Tr}_{E/\mathbb{Q}} \phi_1(x, y)$. Now define $\phi = \zeta^{-1}\phi_1$, so that we have $\psi(x, y) = \operatorname{Tr}_{E/\mathbb{Q}}\bigl(\zeta\phi(x, y)\bigr)$. The uniqueness of ϕ is obvious from the preceding lemma.

It remains to show that we have $\phi(y, x) = \overline{\phi(x, y)}$. Because ψ is antisymmetric, $\psi(y, x) = -\psi(x, y)$, which implies that

$$\operatorname{Tr}_{E/\mathbb{Q}}\bigl(\zeta\phi(y, x)\bigr) = -\operatorname{Tr}_{E/\mathbb{Q}}\bigl(\zeta\phi(x, y)\bigr) = \operatorname{Tr}_{E/\mathbb{Q}}\bigl(\bar{\zeta}\phi(x, y)\bigr).$$

On replacing y by ey, for arbitrary $e \in E$, we obtain

$$\operatorname{Tr}_{E/\mathbb{Q}}\bigl(\zeta e \cdot \phi(y, x)\bigr) = \operatorname{Tr}_{E/\mathbb{Q}}\bigl(\overline{\zeta e} \cdot \phi(x, y)\bigr).$$

On the other hand, we have

$$\operatorname{Tr}_{E/\mathbb{Q}}\bigl(\zeta e \cdot \phi(y, x)\bigr) = \operatorname{Tr}_{E/\mathbb{Q}}\bigl(\overline{\zeta e \cdot \phi(y, x)}\bigr) = \operatorname{Tr}_{E/\mathbb{Q}}\bigl(\overline{\zeta e} \cdot \overline{\phi(y, x)}\bigr).$$

Since $\overline{\zeta e}$ can be an arbitrary element of E, the nondegeneracy of the trace pairing implies that $\phi(x, y) = \overline{\phi(y, x)}$. $\square$

Hodge classes of split Weil type. We will now describe a condition on V that guarantees that the space $\bigwedge_E^d V$ consists entirely of Hodge cycles.

Definition 26.8. Let V be a rational Hodge structure of type $\{(1, 0), (0, 1)\}$ with $E \hookrightarrow \operatorname{End}_{\mathbb{Q}\text{-HS}}(V)$ and $\dim_E V = d$ even. We say that V is *of split Weil type* relative to E if there exists a split E-hermitian form ϕ on V such that $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ defines a polarization on V for some totally imaginary element $\zeta \in E^\times$.

According to Corollary 26.4, the condition on the E-hermitian form ϕ is that there should exist a totally isotropic subspace $W \subseteq V$ with $\dim_E W = d/2$.

Proposition 26.9. *If V is of split Weil type relative to E, then the space*

$$\bigwedge_E^d V \subseteq \bigwedge_{\mathbb{Q}}^d V$$

consists of Hodge classes of type $(d/2, d/2)$.

Proof. For any $s \in S$, let ϕ_S be the induced hermitian form on $V_s = V \otimes_{E,s} \mathbb{C}$. The isomorphism

$$\alpha \colon V \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} V_s$$

respects the Hodge decomposition. According to Lemma 25.5, it suffices to prove that $\dim V_s^{1,0} = \dim V_s^{0,1} = d/2$. We are going to do this by showing that ϕ_s is positive/negative definite on these two subspaces. Since $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ defines a polarization, ϕ is nondegenerate; recall from above that the signature of ϕ_s is (a_s, b_s). Because ϕ is split, Corollary 26.4 shows that we have $a_s = b_s = d/2$ for every embedding $s \in S$. So the signature of ϕ_s is actually $(d/2, d/2)$.

Now let $x \in V_s^{1,0}$ be any nonzero element. Writing $x = \sum_j v_j \otimes z_j$, we have

$$\phi_s(x, x) = \sum_{j,k} z_j \bar{z}_k s\bigl(\phi(v_j, v_k)\bigr).$$

At the same time, the fact that $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ is a polarization tells us that

$$\begin{aligned} \|x\|^2 = i\psi(x, \bar{x}) &= i \sum_{j,k} z_j \bar{z}_k \psi(v_j, v_k) = i \sum_{j,k} z_j \bar{z}_k \operatorname{Tr}_{E/\mathbb{Q}}\bigl(\zeta\phi(v_j, v_k)\bigr) \\ &= i \sum_{j,k} \sum_{s' \in S} z_j \bar{z}_k s'(\zeta) s'\bigl(\phi(v_j, v_k)\bigr) = i \sum_{s' \in S} s'(\zeta)\phi_{s'}(x, x) \end{aligned}$$

is positive. Because $x \in V_s$, the sum is equal to $is(\zeta)\phi_s(x, x)$, and so ϕ_s is either positive or negative definite on $V_s^{1,0}$, depending on the sign of $is(\zeta)$. Because we know the signature of ϕ_s, we get $\dim V_s^{1,0} \leq d/2$. For the same reason, we have $\dim V_s^{0,1} \leq d/2$; but because both dimensions must add up to d, we can then conclude that $\dim V_s^{1,0} = \dim V_s^{0,1} = d/2$. ☐

These special Hodge classes are called *split Weil classes* or more precisely *Hodge classes of split Weil type*. They are the most important examples of Hodge classes on abelian varieties of CM-type; as I said before, the Hodge conjecture is *not* known for these classes except in dimension ≤ 4.

Are there any examples of Hodge structures of split Weil type? Fortunately, there is a simple numerical criterion that can be used to check this. Recall that a CM-type of E is a function $\varphi \colon S \to \{0, 1\}$ with the property that $\varphi(s) + \varphi(\bar{s}) = 1$. It determines a Hodge structure E_φ of CM-type on the $\mathbb{Q}$-vector space E, with Hodge decomposition

$$E_\phi \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), \varphi(\bar{s})}.$$

This is the Hodge structure on $H^1(A, \mathbb{Q})$, where A is a simple abelian variety of CM-type.

Now let $\varphi_1, \ldots, \varphi_d$ be CM-types attached to E. Let $V_i = E_{\varphi_i}$ be the Hodge structure of CM-type corresponding to φ_i, and define

$$V = \bigoplus_{i=1}^d V_i.$$

Then V is a Hodge structure of CM-type with $\dim_E V = d$.

Proposition 26.10. *If $\sum \varphi_i$ is constant on S, then V is of split Weil type.*

Proof. To begin with, it is necessarily the case that $\sum \varphi_i = d/2$; indeed,

$$\sum_{i=1}^{d} \varphi_i(s) + \sum_{i=1}^{d} \varphi(\bar{s}) = \sum_{i=1}^{d} \bigl(\varphi_i(s) + \varphi_i(\bar{s})\bigr) = d,$$

and the two sums are equal by assumption. By construction, we have

$$V \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{i=1}^{d} \bigl(E_{\varphi_i} \otimes_{\mathbb{Q}} \mathbb{C}\bigr) \simeq \bigoplus_{i=1}^{d} \bigoplus_{s \in S} \mathbb{C}^{\varphi_i(s), \varphi_i(\bar{s})}.$$

This shows that

$$V_s = V \otimes_{E,s} \mathbb{C} \simeq \bigoplus_{i=1}^{d} \mathbb{C}^{\varphi_i(s), \varphi_i(\bar{s})}.$$

Therefore $\dim_{\mathbb{C}} V_s^{1,0} = \sum \varphi_i(s) = d/2$, and likewise $\dim_{\mathbb{C}} V_s^{0,1} = \sum \varphi_i(\bar{s}) = d/2$. Of course, this already implies that all classes in $\bigwedge_E^d V$ are Hodge classes.

Next, we construct the required E-hermitian form on V. For each i, choose a Riemann form ψ_i on V_i, whose Rosati involution acts as complex conjugation on E. Since $V_i = E_{\varphi_i}$, there exist totally imaginary elements $\zeta_i \in E^\times$, such that

$$\psi_i(x,y) = \mathrm{Tr}_{E/\mathbb{Q}}\bigl(\zeta_i x \bar{y}\bigr)$$

for every $x, y \in E$. Set $\zeta = \zeta_d$, and define $\phi_i(x,y) = \zeta_i \zeta^{-1} x \bar{y}$, which is an E-hermitian form on V_i with the property that $\psi_i = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta \phi_i)$.

For any collection of totally positive elements $f_i \in F$,

$$\psi = \sum_{i=1}^{d} f_i \psi_i$$

is a Riemann form for V. As E-vector spaces, we have $V = E^{\bigoplus d}$, and so we can define a nondegenerate E-hermitian form on V by the rule

$$\phi(v,w) = \sum_{i=1}^{d} f_i \phi_i(v_i, w_i).$$

We then have $\psi = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta\phi)$. By the same argument as before, $a_s = b_s = d/2$, since $\dim_{\mathbb{C}} V_s^{1,0} = \dim_{\mathbb{C}} V_s^{0,1} = d/2$. By construction, the form ϕ is diagonalized, and so its discriminant is easily found to be

$$\operatorname{disc} \phi = \zeta^{-d} \prod_{i=1}^{d} f_i \zeta_i \mod \mathrm{Nm}_{E/F}(E^\times).$$

On the other hand, we know from general principles that, for any $s \in S$,

$$\operatorname{sgn}\bigl(s(\operatorname{disc}\phi)\bigr) = (-1)^{b_s} = (-1)^{d/2}.$$

This means that $\operatorname{disc}\phi = (-1)^{d/2} f$ for some totally positive element $f \in F^\times$. Upon replacing f_d by $f_d f^{-1}$, we get $\operatorname{disc}\phi = (-1)^{d/2}$, which proves that (V, ϕ) is split. □

26.1. André's theorem and reduction to split Weil classes. The second step in the proof of Deligne's theorem is to reduce the problem from arbitrary Hodge classes on abelian varieties of CM-type to Hodge classes of split Weil type. This is accomplished by the following pretty theorem due to Yves André.

Theorem 26.1 (André)**.** *Let V be a rational Hodge structure of type $\{(1,0),(0,1)\}$, which is of CM-type. Then there exists a CM-field E, rational Hodge structures V_α of split Weil type relative to E, and morphisms of Hodge structure $V_\alpha \to V$, such that every Hodge class $\xi \in \bigwedge_{\mathbb{Q}}^{2p} V$ is a sum of images of Hodge classes $\xi_\alpha \in \bigwedge_{\mathbb{Q}}^{2p} V_\alpha$ of split Weil type.*

Proof. Let $V = V_1 \oplus \cdots \oplus V_r$, with V_i irreducible; then each $E_i = \operatorname{End}_{\mathbb{Q}\text{-HS}}(V_i)$ is a CM-field. Define E to be the Galois closure of the compositum of the fields $E_1, \dots, E_r$. Since V is of CM-type, E is a CM-field which is Galois over $\mathbb{Q}$. Let G be its Galois group over $\mathbb{Q}$. After replacing V by $V \otimes_{\mathbb{Q}} E$ (of which V is a direct factor), we may assume without loss of generality that $E_i = E$ for all i.

As before, let $S = \operatorname{Hom}(E, \mathbb{C})$ be the set of complex embeddings of E; we then have a decomposition

$$V \simeq \bigoplus_{i \in I} E_{\varphi_i}$$

for some collection of CM-types φ_i. Applying Lemma 24.11, we get

$$V \otimes_{\mathbb{Q}} E \simeq \bigoplus_{i \in I} \bigoplus_{g \in G} E_{g\varphi_i}.$$

Since each $E_{g\varphi_i}$ is one-dimensional over E, we get

$$\left(\bigwedge\nolimits_{\mathbb{Q}}^{2p} V\right) \otimes_{\mathbb{Q}} E \simeq \bigwedge\nolimits_{E}^{2p} (V \otimes_{\mathbb{Q}} E) \simeq \bigwedge\nolimits_{E}^{2p} \bigoplus_{(i,g) \in I \times G} E_{g\varphi_i} \simeq \bigoplus_{\substack{\alpha \subseteq I \times G \\ |\alpha| = 2p}} \bigotimes_{(i,g) \in \alpha} E_{g\varphi_i}$$

where the tensor product is over E. If we now define Hodge structures of CM-type

$$V_\alpha = \bigoplus_{(i,g) \in \alpha} E_{g\varphi_i}$$

for any subset $\alpha \subseteq I \times G$ of size $2p$, then V_α has dimension $2p$ over E. The above calculation shows that

$$\left(\bigwedge\nolimits_{\mathbb{Q}}^{2p} V\right) \otimes_{\mathbb{Q}} E \simeq \bigoplus_{\alpha} \bigwedge\nolimits_{E}^{2p} V_\alpha,$$

which is an isomorphism both as Hodge structures and as E-vector spaces. Moreover, as V_α is a sub-Hodge structure of $V \otimes_{\mathbb{Q}} E$, we clearly have morphisms $V_\alpha \to V$, and any Hodge class $\xi \in \bigwedge_{\mathbb{Q}}^{2p} V$ is a sum of Hodge class $\xi_\alpha \in \bigwedge_{E}^{2p} V_\alpha$.

It remains to see that V_α is of split Weil type whenever ξ_α is nonzero. Fix a subset $\alpha \subseteq I \times G$ of size $2p$, with the property that $\xi_\alpha \neq 0$. Note that we have

$$\bigwedge\nolimits_{E}^{2p} V_\alpha \simeq \bigotimes_{(i,g) \in \alpha} E_{g\varphi_i} \simeq E_\varphi,$$

where $\varphi \colon S \to \mathbb{Z}$ is the function

$$\varphi = \sum_{(i,g) \in \alpha} g\varphi_i$$

The Hodge decomposition of E_φ is given by

$$E_\varphi \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), \varphi(\bar{s})}.$$

The image of the Hodge cycle ξ_α in E_φ must be purely of type (p,p) with respect to this decomposition. But

$$\xi_\alpha \otimes 1 \mapsto \sum_{s \in S} s(\xi_\alpha),$$

and since each $s(\xi_\alpha)$ is nonzero (because $\xi_\alpha \neq 0$ and s is an embedding), we conclude that $\varphi(s) = p$ for every $s \in S$. This means that the sum of the $2p$ CM-types $g\varphi_i$, indexed by $(i,g) \in \alpha$, is constant on S. We conclude by the criterion in Proposition 26.10 that V_α is of split Weil type. $\square$

In geometric terms, this is saying that if A is an abelian variety of CM-type, and if $\xi \in H^{2p}(A,\mathbb{Q})$ is a Hodge class, then there are abelian varieties A_α of split Weil type, and morphisms $q_\alpha\colon A \to A_\alpha$, such that $\xi = \sum_\alpha q_\alpha^*(\xi_\alpha)$, where $\xi_\alpha \in H^{2p}(A_\alpha,\mathbb{Q})$ are Hodge classes of split Weil type. So if we can show that all Hodge classes of split Weil type are absolute (or algebraic), then all Hodge classes on abelian varieties of CM-type will also be absolute (or algebraic).