Lecture 23 (April 22)

Deligne's Principle B. The goal of today's lecture is to show that absolute Hodge classes behave well in families. Suppose that $f\colon X \to B$ is a smooth projective morphism (over $\mathbb{C}$); for each $b \in B$, we denote the fiber by $X_b = f^{-1}(b)$, which is a smooth projective variety. For simplicity, let's assume that the parameter space B is connected and quasi-projective; then X itself is also quasi-projective. The $2p$-th cohomology groups $H^{2p}(X_b, \mathbb{Q})$ of the fibers fit together into a local system $R^{2p} f_* \mathbb{Q}_X$ on B. If we have a global section $\alpha \in H^0(B, R^{2p} f_* \mathbb{Q}_X)$, we denote its value at a point $b \in B$ by $\alpha_b \in H^{2p}(X_b, \mathbb{Q})$. We think of α as being a family of cohomology classes on the fibers.

The following important result is known as "Deligne's Principle B". Informally, it says that if we have a family of cohomology classes (in the above sense), and if one of them is an absolute Hodge class, then all of them are absolute Hodge classes. (The analogue problem for algebraic classes is the so-called "variational Hodge conjecture"; this is wide open.)

Theorem 23.1 (Principle B). *Let $f\colon X \to B$ be a smooth projective morphism, with B connected and quasi-projective, and let $\alpha \in H^0(B, R^{2p} f_* \mathbb{Q}_X)$. If there is a point $0 \in B$ such that $\alpha_0 \in H^{2p}(X_0, \mathbb{Q})$ is an absolute Hodge class, then $\alpha_b \in H^{2p}(X_b, \mathbb{Q})$ is an absolute Hodge class for every $b \in B$.*

In practice, this means that if α_0 is the class of an algebraic cycle (and therefore an absolute Hodge classe), then all the α_b are absolute Hodge classes. So Principle B allows us to bypass the Hodge conjecture in certain cases.

Properties of absolute Hodge classes. We are going to prove the theorem by studying the behavior of absolute Hodge classes under various operations.

Pullbacks. The most basic operation is pulling back along a morphism $f\colon X \to Y$ between two smooth projective varieties (over $\mathbb{C}$). Here the pullback morphism

$$f^*\colon H^k(Y, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

takes absolute Hodge classes on Y to absolute Hodge classes on X. This can be seen as follows. First, we have a pullback morphism in algebraic de Rham cohomology: the morphism of sheaves $f^*\Omega^1_{Y/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}}$ induces a morphism of complexes

$$f^*\Omega^\bullet_{Y/\mathbb{C}} \to \Omega^\bullet_{X/\mathbb{C}}$$

between the algebraic de Rham complexes of X and Y; passing to cohomology gives

$$H^k_{dR}(Y/\mathbb{C}) = H^k(Y, \Omega^\bullet_{Y/\mathbb{C}}) \to H^k(X, f^*\Omega^\bullet_{Y/\mathbb{C}}) \to H^k(X, \Omega^\bullet_{X/\mathbb{C}}) = H^k_{dR}(X/\mathbb{C}).$$

It is easy to see that this morphism is compatible with $f^*\colon H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$ under the comparison isomorphism with algebraic de Rham cohomology. Now if $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, then we get a conjugate morphism $f^\sigma\colon X^\sigma \to Y^\sigma$, and the compatibility with algebraic de Rham cohomology implies that

$$f^*(\alpha^\sigma) = (f^*\alpha)^\sigma$$

So if $\alpha \in H^k(Y, \mathbb{Q})$ is an absolute Hodge class (and k is even), then $\alpha^\sigma \in H^k(Y^\sigma, \mathbb{Q})$, and so its pullback lies in $H^k(X^\sigma, \mathbb{Q})$, which shows that $f^*\alpha$ is again an absolute Hodge class.

Cup product. Similarly, the cup product morphism

$$H^i(X,\mathbb{Q}) \otimes H^j(X,\mathbb{Q}) \to H^{i+j}(X,\mathbb{Q}), \quad \alpha \otimes \beta \mapsto \alpha \cup \beta,$$

takes pairs of absolute Hodge classes to absolute Hodge classes. To see this, we rewrite the cup product as

$$H^i(X,\mathbb{Q}) \otimes H^j(X,\mathbb{Q}) \hookrightarrow H^{i+j}(X \times X,\mathbb{Q}) \xrightarrow{\Delta^*} H^{i+j}(X,\mathbb{Q}),$$

where the first morphism comes from the Künneth isomorphism, and the second is pullback along the diagonal $\Delta\colon X \to X \times X$. The Künneth isomorphism also holds in algebraic de Rham cohomology, in a way that is compatible with the comparison isomorphism; this is a consequence of the fact that

$$\Omega^1_{X\times X/\mathbb{C}} \cong p_1^*\Omega^1_{X/\mathbb{C}} \otimes p_2^*\Omega^1_{X/\mathbb{C}}.$$

For that reason, the inclusion $H^i(X,\mathbb{Q}) \otimes H^j(X,\mathbb{Q}) \hookrightarrow H^{i+j}(X \times X,\mathbb{Q})$ takes a pair of absolute Hodge classes to an absolute Hodge class; and because Δ^* preserves absolute Hodge classes, we get the result.

Poincaré duality. On a smooth projective variety X of dimension n, the pairing

$$H^k(X,\mathbb{Q}) \otimes H^{2n-k}(X,\mathbb{Q}) \to H^{2n}(X,\mathbb{Q}), \quad \alpha \otimes \beta \mapsto \alpha \cup \beta,$$

is nondegenerate, which means that

$$H^k(X,\mathbb{Q}) \to \operatorname{Hom}\bigl(H^{2n-k}(X,\mathbb{Q}), H^{2n}(X,\mathbb{Q})\bigr)$$

is an isomorphism. As Hodge structures of weight $2n$, we have $H^{2n}(X,\mathbb{Q}) \cong \mathbb{Q}(-n)$; an explicit isomorphism is given by

$$H^{2n}(X,\mathbb{Q}) \to \mathbb{Q}(-n), \quad \alpha \mapsto \frac{1}{(2\pi i)^n}\int_X \alpha.$$

Its inverse is represented by the fundamental class $[x] \in H^{2n}(X,\mathbb{Q}(n))$ of any point $x \in X$. Because cup product preserves absolute Hodge classes, and because the fundamental class of a point is of course an absolute Hodge class, it follows that the Poincaré duality isomorphism

$$H^k(X,\mathbb{Q}) \to \operatorname{Hom}\bigl(H^{2n-k}(X,\mathbb{Q}), \mathbb{Q}(-n)\bigr)$$

takes absolute Hodge classes to absolute Hodge classes. (The notion of absolute Hodge classes also makes sense for classes in the dual vector space.)

Example 23.2. Let $f\colon X \to Y$ be a morphism between smooth projective varieties, and set $r = \dim Y - \dim X$. Then the Gysin homomorphism $f_*\colon H^k(X,\mathbb{Q}) \to H^{k+2r}(Y,\mathbb{Q}(r))$ takes absolute Hodge classes to absolute Hodge classes. The reason is that f_* is the composition of Poincaré duality on X and Y and the homomorphism dual to $f^*\colon H^k(Y,\mathbb{Q}) \to H^k(X,\mathbb{Q})$.

Absolute homomorphisms. More generally, suppose that we have a homomorphism $\phi\colon H^k(X,\mathbb{Q}) \to H^{k+2r}(Y,\mathbb{Q}(r))$ between cohomology groups of two smooth projective varieties X and Y. The Tate twist changes the weight of the second cohomology group to $k+2r-2r = k$. Using Poincaré duality and the Künneth formula, we can associate to ϕ a cohomology class $\operatorname{cl}(\phi)$ in

$$\begin{aligned} H^k(X,\mathbb{Q})^\vee \otimes H^{k+2r}(Y,\mathbb{Q}(r)) &\cong H^{2n-k}(X,\mathbb{Q}(n)) \otimes H^{k+2r}(Y,\mathbb{Q}(r)) \\ &\subseteq H^{2n+2r}(X \times Y,\mathbb{Q}(n+r)), \end{aligned}$$

where $n = \dim X$. It is not hard to see that ϕ is a morphism of Hodge structures of weight k if and only if $\operatorname{cl}(\phi)$ is a Hodge class on $X \times Y$.

Definition 23.3. We will say that a morphism $\phi\colon H^k(X,\mathbb{Q}) \to H^{k+2r}(Y,\mathbb{Q}(r))$ is *absolute* if $\operatorname{cl}(\phi) \in H^{2n+2r}(X \times Y,\mathbb{Q}(n+r))$ is an absolute Hodge class.

Example 23.4. The Gysin homomorphism is absolute.

One can recover the action of ϕ by a formula similar to an integral transform:

$$\phi(\alpha) = (p_2)_* \bigl(p_1^*(\alpha) \cup \operatorname{cl}(\phi)\bigr),$$

with $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ the two projections. This shows that if ϕ is absolute, then it takes absolute Hodge classes in $H^k(X, \mathbb{Q})$ to absolute Hodge classes in $H^{k+2r}(Y, \mathbb{Q}(r))$ (when k is even). Indeed, all three operations on the right-hand side of the formula preserve absolute Hodge classes.

Composition and inverses. The composition of absolute morphisms is absolute. For simplicity, let's take the case where $\phi \colon H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})$ and $\psi \colon H^k(Y, \mathbb{Q}) \to H^k(Z, \mathbb{Q})$ are homomorphisms between cohomology groups of the same degree. The associated cohomology classes are $\operatorname{cl}(\phi) \in H^{2n}(X \times Y, \mathbb{Q}(n))$ and $\operatorname{cl}(\psi) \in H^{2m}(Y \times Z, \mathbb{Q}(m))$, where $n = \dim X$ and $m = \dim Y$. Just as with integral transforms, the cohomology class of the composition $\psi \circ \phi$ is computed by a convolution:

$$\operatorname{cl}(\psi \circ \phi) = (p_{13})_* \bigl(p_{12}^* \operatorname{cl}(\phi) \cup p_{23}^* \operatorname{cl}(\psi)\bigr) \in H^{2n}(X \times Z, \mathbb{Q}(n)).$$

If $\operatorname{cl}(\phi)$ and $\operatorname{cl}(\psi)$ are absolute Hodge classes, then so is their convolution; therefore $\psi \circ \phi$ is again absolute. Similarly, one shows that if

$$\phi \colon H^k(X, \mathbb{Q}) \to H^{k+2r}(Y, \mathbb{Q}(r))$$

is both absolute and an isomorphism, then the inverse homomorphism ϕ^{-1} is again absolute.

Images of absolute morphisms. We'll now use the facts from the previous section to prove the following result.

Proposition 23.5. *Let X and Y be smooth projective varieties. Suppose that $\phi \colon H^{2p}(X, \mathbb{Q}) \to H^{2p}(Y, \mathbb{Q})$ is an absolute morphism of Hodge structures. If $\alpha \in H^{2p}(Y, \mathbb{Q})$ is an absolute Hodge class in the image of ϕ, then there is an absolute Hodge class $\beta \in H^{2p}(X, \mathbb{Q})$ such that $\phi(\beta) = \alpha$.*

In other words, any absolute Hodge class in the image of an absolute morphism is actually the image of an absolute Hodge class. The proof relies on the fact that the two Hodge structures can be polarized, in a way that is compatible with absolute Hodge classes.

Let's start with a few general remarks. Let H be a Hodge structure of weight k, with Hodge decomposition

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}.$$

Define the Weil operator $C \in \operatorname{End}(H_{\mathbb{R}})$ by the formula $Cv = i^{p-q} v$ for $v \in H^{p,q}$. By Hodge symmetry, we have $C(\bar{v}) = \overline{Cv}$, and so C is a real operator with $C^2 = (-1)^k$. Now recall that a polarization is a $(-1)^k$-symmetric pairing

$$S \colon H \otimes_{\mathbb{Q}} H \to \mathbb{Q}(-k)$$

such that $\langle v, w \rangle = S(Cv, \bar{w})$ is a hermitian inner product on $H_{\mathbb{C}}$ that makes the Hodge decomposition into an orthogonal decomposition. This implies in particular that the polarization S is non-degenerate: if $S(v, w) = 0$ for all $w \in H_{\mathbb{C}}$, then $\|v\|^2 = S(Cv, \bar{v}) = 0$, and so $v = 0$. If we consider S as a homomorphism

$$S \colon H \to \operatorname{Hom}_{\mathbb{Q}}\bigl(H, \mathbb{Q}(-k)\bigr),$$

it is therefore an isomorphism of Hodge structures (of weight k).

If $V \subseteq H$ is a sub-Hodge structure, meaning a rational subspace such that $V^{p,q} = H^{p,q} \cap V_{\mathbb{C}}$, then the orthogonal complement

$$V^{\perp} = \bigl\{\, h \in H \bigm| S(h, v) = 0 \text{ for all } v \in V \,\bigr\}$$

is again a sub-Hodge structure, and $H = V \oplus V^\perp$. This follows from the fact that

$$V^\perp \otimes_{\mathbb{Q}} \mathbb{C} = \big\{ h \in H_{\mathbb{C}} \bigm| \langle h, v \rangle = 0 \text{ for all } v \in V_{\mathbb{C}} \big\},$$

which holds because $\langle v, w \rangle = S(Cv, \bar{w})$ and because the Hodge decomposition is orthogonal with respect to the inner product.

In the geometric case, the polarization is itself absolute, in the sense we talked about earlier. Let's recall the construction; to keep down the notation, I am going to leave out the Tate twists in the formulas below. Let X be a smooth projective variety of dimension n, choose an ample line bundle $L \in \operatorname{Pic}(X)$, and let $\omega = c_1(L) \in H^2(X, \mathbb{Z}(1))$ be its first Chern class. The pairing

$$S_k(\alpha, \beta) = \frac{(-1)^{k(k-1)/2}}{(2\pi i)^n} \int_X \alpha \cup \beta \cup \omega^{n-k},$$

takes values in $\mathbb{Q}(-k)$, and is a polarization of the Hodge structure on the primitive cohomology

$$H_0^k(X, \mathbb{Q}) = \ker\big(\omega^{n-k+1} \colon H^k(X, \mathbb{Q}) \to H^{2n-k+2}(X, \mathbb{Q})\big)\big).$$

The formula for S_k only involves absolute operations: $\omega = c_1(L)$ is an absolute Hodge class, and the isomorphism

$$H^{2n}(X, \mathbb{Q}) \to \mathbb{Q}(-n), \quad \alpha \mapsto \frac{1}{(2\pi i)^n} \int_X \alpha,$$

is the inverse of the fundamental class of a point.

To get a polarization on all of $H^k(X, \mathbb{Q})$, we use the Lefschetz decomposition

$$H^k(X, \mathbb{Q}) = H_0^k(X, \mathbb{Q}) \oplus \omega H_0^{k-2}(X, \mathbb{Q}) \oplus \omega^2 H_0^{k-4}(X, \mathbb{Q}) \oplus \cdots$$

Define an involution $s \in \operatorname{End} H^k(X, \mathbb{Q})$ by acting as $(-1)^\ell$ on the subspace $\omega^\ell H_0^{k-2\ell}(X, \mathbb{Q})$ in the Lefschetz decomposition. Then

$$S(\alpha, \beta) = S_k\big(\alpha, s(\beta)\big)$$

polarizes the Hodge structure on $H^k(X, \mathbb{Q})$. As we said above, we can view S as an isomorphism

$$S \colon H^k(X, \mathbb{Q}) \to \operatorname{Hom}\big(H^k(X, \mathbb{Q}), \mathbb{Q}(-k)\big), \tag{23.6}$$

and this isomorphism is absolute; we'll abbreviate this by saying that the polarization is absolute.

Proposition 23.7. *The isomorphism in* (23.6) *is absolute.*

Proof. Because the formula for S_k only involves absolute operations, it suffices to prove that the involution s is absolute. Let $p_\ell \colon H^k(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$ be the projection to the subspace $\omega^\ell H_0^{k-2\ell}(X, \mathbb{Q})$ in the Lefschetz decomposition. Then

$$s = \sum_{\ell \in \mathbb{N}} (-1)^\ell p_\ell,$$

and so it is enough to prove that each p_ℓ is absolute. Take any $\alpha \in H^k(X, \mathbb{Q})$, and write its Lefschetz decomposition as

$$\alpha = \alpha_0 + \omega \cup \alpha_1 + \omega^2 \cup \alpha_2 + \cdots.$$

Here each $\alpha_\ell \in H_0^{k-2\ell}(X, \mathbb{Q})$ is primitive, which means that $\omega^{n-k+2\ell} \cup \alpha_\ell \neq 0$ and $\omega^{n-k+2\ell+1} \cup \alpha_\ell = 0$. For $r \geq 1$, we therefore have

$$\omega^{n-k+r} \cup \alpha = \sum_{\ell \geq r} \omega^{n-k+r+\ell} \cup \alpha_\ell \in H^{2n-k+2r}(X, \mathbb{Q}).$$

By the Hard Lefschetz theorem,

$$\omega^{n-k+2r} \colon H^{k-2r}(X, \mathbb{Q}) \to H^{2n-k+2r}(X, \mathbb{Q})$$

is an isomorphism, and we clearly have

$$(\omega^{n-k+2r})^{-1}\big(\omega^{n-k+r}\cup\alpha\big)=\sum_{\ell\geq r}\omega^{\ell-r}\alpha_\ell.$$

By comparing this with the original Lefschetz decomposition for α, we find that

$$(p_0+\cdots+p_{r-1})(\alpha)=\alpha-\sum_{\ell\geq r}\omega^\ell\cup\alpha_\ell=\alpha-\omega^r\cup(\omega^{n-k+2r})^{-1}\big(\omega^{n-k+r}\cup\alpha\big).$$

This is clearly an absolute morphism, because it only involves cup product with the absolute Hodge class ω and the inverse of the absolute isomorphism ω^{n-k+2r}. By subtracting the formulas for r and $r+1$, we conclude that each projector p_r is absolute. □

We can now show that if an absolute Hodge class lies in the image of an absolute morphism, then it must be the image of an absolute Hodge class.

Proof of Proposition 23.5. Let's denote by S_X and S_Y the polarizations on $H^{2p}(X,\mathbb{Q})$ and $H^{2p}(Y,\mathbb{Q})$. The absolute morphism $\phi\colon H^{2p}(X,\mathbb{Q})\to H^{2p}(Y,\mathbb{Q})$ has an adjoint $\phi^\dagger\colon H^{2p}(Y,\mathbb{Q})\to H^{2p}(X,\mathbb{Q})$ with respect to the polarizations, which satisfies

$$S_Y\big(\alpha,\phi(\beta)\big)=S_X\big(\phi^\dagger(\alpha),\beta)\big).$$

The adjoint fits into a commutative diagram

$$\begin{tikzcd}
H^{2p}(Y,\mathbb{Q}) \arrow[r, "\phi^\dagger"] \arrow[d, "S_Y"] & H^{2p}(X,\mathbb{Q}) \arrow[d, "S_X"] \\
\operatorname{Hom}\big(H^{2p}(Y,\mathbb{Q}),\mathbb{Q}(-p)\big) \arrow[r, "\phi^*"] & \operatorname{Hom}\big(H^{2p}(X,\mathbb{Q}),\mathbb{Q}(-p)\big),
\end{tikzcd}$$

where ϕ^* is the morphism induced by ϕ. Because ϕ is absolute, the dual morphism ϕ^* is also absolute; and because S_X is absolute, its inverse S_X^{-1} is also absolute. Therefore $\phi^\dagger$ is absolute as well. Note that $\phi^\dagger$ is also the adjoint of ϕ with respect to the inner products on $H^{2p}(X,\mathbb{Q})$ and $H^{2p}(Y,\mathbb{Q})$.

Because ϕ is a morphism of Hodge structures, the polarization S_Y gives us an orthogonal decomposition

$$H^{2p}(Y,\mathbb{Q})=\operatorname{im}\phi\oplus(\operatorname{im}\phi)^\perp.$$

Just as in linear algebra, the adjoint has the property that $(\operatorname{im}\phi)^\perp=\ker\phi^\dagger$. We can therefore rewrite the decomposition as

$$H^{2p}(Y,\mathbb{Q})=\operatorname{im}\phi\oplus\ker\phi^\dagger.$$

Now consider the morphism $\phi\circ\phi^\dagger\colon H^{2p}(Y,\mathbb{Q})\to H^{2p}(Y,\mathbb{Q})$. It is self-adjoint, and its kernel is exactly $\ker\phi^\dagger$, because of the identity

$$\big\langle\alpha,(\phi\circ\phi^\dagger)(\alpha)\big\rangle_Y=\big\langle\phi^\dagger(\alpha),\phi^\dagger(\alpha)\big\rangle_X.$$

Let $\pi\colon H^{2p}(Y,\mathbb{Q})\to H^{2p}(Y,\mathbb{Q})$ denote the orthogonal projection to the subspace $\operatorname{im}\phi$. By the spectral theorem (for self-adjoint linear operators), π can be written as a polynomial in $\phi\circ\phi^\dagger$ without constant term, say

$$\pi=\sum_{n\geq 1}c_n(\phi\circ\phi^\dagger)^n.$$

Now if $\alpha\in H^{2p}(Y,\mathbb{Q})$ is an absolute Hodge class in the image of ϕ, then

$$\alpha=\pi(\alpha)=\sum_{n\geq 1}c_n(\phi\circ\phi^\dagger)^n\alpha,$$

which is equal to the image under ϕ of the absolute Hodge class

$$\beta = \sum_{n \geq 1} c_n (\phi^\dagger \circ \phi)^{n-1} \phi^\dagger(\alpha) \in H^{2p}(X, \mathbb{Q}).$$

This proves the proposition. □

Proof of Principle B. After all this work, it is now an easy matter to prove Deligne's Principle B. Let $f\colon X \to B$ be a smooth projective morphism, with B connected and quasi-projective. Let $\alpha \in H^0(B, R^{2p} f_* \mathbb{Q})$ be a section of the local system, and denote by $\alpha_b \in H^{2p}(X_b, \mathbb{Q})$ its value at a point $b \in B$. Suppose that $\alpha_0 \in H^{2p}(X_0, \mathbb{Q})$ is an absolute Hodge class for some $0 \in B$. The local system contains the same information as the monodromy action of $\pi_1(B, 0)$ on the cohomology group $H^{2p}(X_0, \mathbb{Q})$, and a global section is the same as a cohomology class that is invariant under monodromy:

$$H^0(B, R^{2p} f_* \mathbb{Q}) \cong H^{2p}(X_0, \mathbb{Q})^{\pi_1(B,0)}$$

From the Leray spectral sequence (which degenerates at E_2 because f is smooth and projective), we get a surjection

$$H^{2p}(X, \mathbb{Q}) \to H^0(B, R^{2p} f_* \mathbb{Q}) \cong H^{2p}(X_0, \mathbb{Q})^{\pi_1(B,0)}.$$

Denoting by $i_b\colon X_b \hookrightarrow X$ the inclusion of the fiber, the composition is just i_0^*.

Now let $\bar{X}$ be a smooth projective variety containing X as a Zariski-open subset, and let $j\colon X \hookrightarrow \bar{X}$ be the open embedding. According to the global invariant cycle theorem, the composition

$$H^{2p}(\bar{X}, \mathbb{Q}) \xrightarrow{j^*} H^{2p}(X, \mathbb{Q}) \xrightarrow{i_0^*} H^{2p}(X_0, \mathbb{Q})^{\pi_1(B,0)}$$

is surjective. (This theorem is also due to Deligne; it uses the fact that $H^{2p}(X, \mathbb{Q})$ has a mixed Hodge structure with weights $\geq 2p$, and that the part of weight $2p$ is exactly the image of j^*.) Our absolute Hodge class $\alpha_0 \in H^{2p}(X_0, \mathbb{Q})$ is invariant under monodromy (because it comes from a global section α), and so it belongs to the image; by Proposition 23.5, it is the image of an absolute Hodge class $\beta \in H^{2p}(\bar{X}, \mathbb{Q})$. But then we have

$$\alpha_b = i_b^* j^* \beta$$

for every $b \in B$, due to the fact that B is connected; and this shows that each α_b is an absolute Hodge class.