Lecture 22: April 29

Today, I would like to discuss a very useful class of examples, namely regular holonomic $\mathscr{D}$-modules of "normal crossing type". We will show that these objects have a simple combinatorial description in terms of vector spaces and certain linear maps between them. We will describe them both on affine space and on projective space. Before we can do that, we need to review a few basic results about $\mathscr{D}$-modules on projective space.

$\mathscr{D}$-affine varieties. We have already seen that algebraic $\mathscr{D}$-modules on affine space are the same thing as modules over the Weyl algebra $A_n(k)$. Somewhat surprisingly, a similar result holds on projective space. In fact, projective space turns out to be $\mathscr{D}$-affine, in the following sense.

Definition 22.1. A nonsingular algebraic variety X is called *$\mathscr{D}$-affine* if it satisfies the following two conditions:

(a) The global section functor
$$\Gamma(X, -)\colon \operatorname{Mod}_{qc}(\mathscr{D}_X) \to \operatorname{Mod}\big(\Gamma(X, \mathscr{D}_X)\big)$$
is exact.

(b) If $\Gamma(X, \mathcal{M}) = 0$ for some $\mathcal{M} \in \operatorname{Mod}_{qc}(\mathscr{D}_X)$, then $\mathcal{M} = 0$.

Here $\operatorname{Mod}_{qc}(\mathscr{D}_X)$ denotes the category of left $\mathscr{D}_X$-modules that are quasi-coherent as $\mathscr{O}_X$-modules; earlier on, we used the term "algebraic $\mathscr{D}$-modules".

Example 22.2. Any nonsingular affine variety is $\mathscr{D}$-affine; in fact, the global sections functor is exact on all quasi-coherent $\mathscr{O}_X$-modules in that case.

Suppose that $\mathcal{M}$ is a left $\mathscr{D}_X$-module. The space of global sections $\Gamma(X, \mathcal{M})$ is then naturally a left module over the ring of global differential operators $\Gamma(X, \mathscr{D}_X)$. On a $\mathscr{D}$-affine variety, this gives an equivalence of categories between algebraic $\mathscr{D}$-modules and modules over the ring $\Gamma(X, \mathscr{D}_X)$.

Theorem 22.3. *Let X be a nonsingular algebraic variety that is $\mathscr{D}$-affine.*

(1) Any $\mathcal{M} \in \operatorname{Mod}_{qc}(\mathscr{D}_X)$ is generated by its global sections.

(2) The global sections functor
$$\Gamma(X, -)\colon \operatorname{Mod}_{qc}(\mathscr{D}_X) \to \operatorname{Mod}\big(\Gamma(X, \mathscr{D}_X)\big)$$
is an equivalence of categories, with inverse $\mathscr{D}_X \otimes_{\Gamma(X, \mathscr{D}_X)} (-)$.

Proof. To simplify the notation, set $R = \Gamma(X, \mathscr{D}_X)$. For (1), we need to show that the natural morphism $\mathscr{D}_X \otimes_R \Gamma(X, \mathcal{M}) \to \mathcal{M}$ is surjective. Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the image. Since the global sections functor is exact by (a), we get a short exact sequence
$$0 \to \Gamma(X, \mathcal{M}_0) \to \Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{M}/\mathcal{M}_0) \to 0.$$
The first two spaces are equal by construction, and so $\Gamma(X, \mathcal{M}/\mathcal{M}_0) = 0$, from which it follows by (b) that $\mathcal{M}_0 = \mathcal{M}$. This proves (1).

Now we turn to (2). The claim is that the inverse functor is given by sending a left $\Gamma(X, \mathscr{D}_X)$-module V to the left $\mathscr{D}_X$-module $\mathscr{D}_X \otimes_R V$. It suffices to show that the two natural morphisms
$$\alpha_{\mathcal{M}}\colon \mathscr{D}_X \otimes_R \Gamma(X, \mathcal{M}) \to \mathcal{M}$$
$$\beta_V\colon V \to \Gamma(X, \mathscr{D}_X \otimes_R V)$$
are isomorphisms for every $\mathcal{M} \in \operatorname{Mod}_{qc}(\mathscr{D}_X)$ and every $V \in \operatorname{Mod}(R)$. Let us first prove that β_V is an isomorphism. This is clearly the case when V is a direct sum of copies of R. When V is an arbitrary R-module, we choose a presentation
$$R^{\oplus I} \longrightarrow R^{\oplus J} \longrightarrow V \longrightarrow 0$$

where I and J are two (possibly infinite) sets. We then get the following diagram with exact rows:

$$\begin{tikzcd}
R^{\oplus I} \arrow[r] \arrow[d, "\cong"] & R^{\oplus J} \arrow[r] \arrow[d, "\cong"] & V \arrow[r] \arrow[d, "\beta_V"] & 0 \\
\Gamma(X, \mathscr{D}_X^{\oplus I}) \arrow[r] & \Gamma(X, \mathscr{D}_X^{\oplus J}) \arrow[r] & \Gamma(X, \mathscr{D}_X \otimes_R V) \arrow[r] & 0
\end{tikzcd}$$

The bottom row is exact because tensor product is right-exact, and because the global sections functor is exact by condition (a) in the definition. Now the 5-lemma implies that β_V is an isomorphism.

It remains to show that $\alpha_{\mathcal{M}}$ is an isomorphism. We already know that $\alpha_{\mathcal{M}}$ is surjective; setting $\mathcal{K} = \ker \alpha_{\mathcal{M}}$, we have a short exact sequence of $\mathscr{D}_X$-modules

$$0 \to \mathcal{K} \to \mathscr{D}_X \otimes_R \Gamma(X, \mathcal{M}) \to \mathcal{M} \to 0$$

and therefore, again by (a), a short exact sequence of R-modules

$$0 \to \Gamma(X, \mathcal{K}) \to \Gamma\big(X, \mathscr{D}_X \otimes_R \Gamma(X, \mathcal{M})\big) \xrightarrow{\beta} \Gamma(X, \mathcal{M}) \to 0.$$

Since we have already shown that $\beta = \beta_{\Gamma(X,\mathcal{M})}$ is an isomorphism, it follows that $\Gamma(X, \mathcal{K}) = 0$, and hence by (b) that $\mathcal{K} = 0$. This concludes the proof of (2). □

As you would expect, coherent $\mathscr{D}_X$-modules correspond to finitely generated $\Gamma(X, \mathscr{D}_X)$-modules.

Corollary 22.4. *If X is $\mathscr{D}$-affine, then*

$$\Gamma(X, -)\colon \operatorname{Mod}_{coh}(\mathscr{D}_X) \to \operatorname{Mod}_{fg}\big(\Gamma(X, \mathscr{D}_X)\big)$$

is also an equivalence of categories.

Proof. We keep the notation $R = \Gamma(X, \mathscr{D}_X)$. If V is a finitely generated R-module, then $\mathscr{D}_X \otimes_R V$ is clearly a coherent $\mathscr{D}_X$-module. Thus we only have to show that $\Gamma(X, \mathcal{M})$ is a finitely generated R-module whenever $\mathcal{M} \in \operatorname{Mod}_{coh}(\mathscr{D}_X)$. Concretely, we have to find finitely many global sections that generate $\mathcal{M}$ as a $\mathscr{D}_X$-module.

Since $\mathcal{M}$ is coherent, the restriction of $\mathcal{M}$ to any affine open subset $U \subseteq X$ is generated as a $\mathscr{D}_U$-module by finitely many sections in $\Gamma(U, \mathcal{M})$. The isomorphism $\mathscr{D}_X \otimes_R \Gamma(X, \mathcal{M}) \cong \mathcal{M}$ in the theorem gives

$$\Gamma(U, \mathscr{D}_X) \otimes_R \Gamma(X, \mathcal{M}) \cong \Gamma(U, \mathcal{M}),$$

and so $\mathcal{M}\big|_U$ is generated as a $\mathscr{D}_U$-module by finitely many sections in $\Gamma(X, \mathcal{M})$. Now X is quasi-compact, hence covered by finitely many affine open subsets; it follows that finitely many global sections generate $\mathcal{M}$ as a $\mathscr{D}_X$-module. In other words, we have a surjective morphism

$$\mathscr{D}_X^{\oplus r} \to \mathcal{M} \to 0.$$

Because the global sections functor is exact by (a), we get a surjection

$$R^{\oplus r} = \Gamma(X, \mathscr{D}_X^{\oplus r}) \to \Gamma(X, \mathcal{M}) \to 0,$$

and so $\Gamma(X, \mathcal{M})$ is a finitely generated R-module. □

We are now going to show that projective spaces are $\mathscr{D}$-affine.

Theorem 22.5. *The projective space $\mathbb{P}_k^n$ is $\mathscr{D}$-affine.*

Proof. Let me begin with a preliminary discussion about global sections on $\mathbb{P}^n$. On $\mathbb{A}^{n+1}$, we have coordinates $x_0, x_1, \dots, x_n$. Let $X \subseteq \mathbb{A}^{n+1}$ be the open complement of the origin. Then $\mathbb{P}^n$ is the quotient of X by the $\mathbb{G}_m$-action that rescales the coordinates. We denote the quotient morphism by $\pi\colon X \to \mathbb{P}_k^n$; the open embedding by $j\colon X \hookrightarrow \mathbb{A}^{n+1}$; and the closed embedding of the origin by $i\colon \operatorname{Spec} k \hookrightarrow \mathbb{A}^{n+1}$. Here are the three morphisms in diagram form:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "\pi"] & \mathbb{A}^{n+1} & \operatorname{Spec} k \arrow[l, hook', "i"'] \\ \mathbb{P}^n \end{tikzcd}$$

The Euler vector field $\theta = x_0\partial_0 + x_1\partial_1 + \dots + x_n\partial_n$ is tangent to the fibers of π. Now suppose that $\mathcal{M}$ is a left $\mathscr{D}_{\mathbb{P}^n}$-module. Then $\mathbb{G}_m$ acts on the space of global sections of $\pi^*\mathcal{M} = \mathscr{O}_X \otimes_{\pi^{-1}\mathscr{O}_{\mathbb{P}^n}} \pi^{-1}\mathcal{M}$, and this gives us a direct sum decomposition

$$\Gamma(X, \pi^*\mathcal{M}) = \bigoplus_{\ell\in\mathbb{Z}} \Gamma_\ell(X, \pi^*\mathcal{M});$$

here $\mathbb{G}_m$ acts on the subspace $\Gamma_\ell(X, \pi^*\mathcal{M})$ with the character $z \mapsto z^\ell$. It follows that θ operates on $\Gamma_\ell(X, \pi^*\mathcal{M})$ as multiplication by ℓ. We have

$$\Gamma(\mathbb{P}^n, \mathcal{M}) \cong \Gamma(X, \pi^*\mathcal{M})^{\mathbb{G}_m} = \Gamma_0(X, \pi^*\mathcal{M}); \tag{22.6}$$

indeed, pullbacks of global sections from $\mathbb{P}^n$ are clearly $\mathbb{G}_m$-invariant, and conversely, any $\mathbb{G}_m$-invariant section on X descends to a global section on $\mathbb{P}^n$. Also note that multiplication by x_j takes Γ_ℓ into $\Gamma_{\ell+1}$, and multiplication by ∂_j takes Γ_ℓ into $\Gamma_{\ell-1}$; the reason is that $[\theta, x_j] = x_j$ and $[\theta, \partial_j] = -\partial_j$.

Now let us start proving that $\mathbb{P}^n$ satisfies the two conditions in (a) and (b). We first show that the global sections functor is exact. Let

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

be a short exact sequence of quasi-coherent $\mathscr{D}_{\mathbb{P}^n}$-modules. Since π is smooth, the pullback functor π^* is exact, which means that

$$0 \to \pi^*\mathcal{M}_1 \to \pi^*\mathcal{M}_2 \to \pi^*\mathcal{M}_3 \to 0$$

is a short exact sequence of quasi-coherent $\mathscr{D}_X$-modules. Because $j\colon X \hookrightarrow \mathbb{A}^{n+1}$ is an open embedding, $j_+ \cong \mathbf{R}j_*$ (after the appropriate conversion between left and right $\mathscr{D}$-modules). Thus we get an exact sequence of quasi-coherent $\mathscr{D}_{\mathbb{A}^{n+1}}$-modules

$$0 \to j_*\pi^*\mathcal{M}_1 \to j_*\pi^*\mathcal{M}_2 \to j_*\pi^*\mathcal{M}_3 \to R^1 j_*\pi^*\mathcal{M}_1 \to \cdots$$

The global sections functor on the affine space $\mathbb{A}^{n+1}$ is exact, and so we finally obtain an exact sequence of A_{n+1}-modules

$$0 \to \Gamma(X, \pi^*\mathcal{M}_1) \to \Gamma(X, \pi^*\mathcal{M}_2) \to \Gamma(X, \pi^*\mathcal{M}_3) \to \Gamma\big(\mathbb{A}^{n+1}, R^1 j_*\pi^*\mathcal{M}_1\big) \to \cdots$$

Now $R^1 j_*\pi^*\mathcal{M}_1$ is a quasi-coherent $\mathscr{D}_{\mathbb{A}^{n+1}}$-module supported on the origin, and so by Kashiwara's equivalence (from Lecture 13), it must be the direct image of a quasi-coherent $\mathscr{D}_{\operatorname{Spec} k}$-module. Concretely, we have

$$\Gamma\big(\mathbb{A}^{n+1}, R^1 j_*\pi^*\mathcal{M}_1\big) \cong k[\partial_0, \partial_1, \dots, \partial_n] \otimes_k V,$$

where V is a k-vector space. The key point is now that θ acts on the right-hand side with strictly negative eigenvalues. Indeed, for any $\alpha \in \mathbb{N}^{n+1}$, we have

$$\theta \cdot \partial^\alpha \otimes v = \sum_{j=0}^n x_j\partial_j \cdot \partial^\alpha \otimes v = \sum_{j=0}^n -(\alpha_j + 1)\partial^\alpha \otimes v = -\big(|\alpha| + n + 1\big) \cdot \partial^\alpha \otimes v.$$

The conclusion is that

$$0 \to \Gamma_0(X, \pi^*\mathcal{M}_1) \to \Gamma_0(X, \pi^*\mathcal{M}_2) \to \Gamma_0(X, \pi^*\mathcal{M}_3) \to 0$$

is short exact; because of (22.6), this proves that $\Gamma(\mathbb{P}^n, -)$ is an exact functor.

All that is left is to show that $\Gamma(\mathbb{P}^n, \mathcal{M}) = 0$ implies $\mathcal{M} = 0$. Here we argue by contradiction and assume that $\mathcal{M} \neq 0$. Since $\pi \colon X \to \mathbb{P}^n$ has a section over each of the $n+1$ basic affine open subsets, we must have $\pi^*\mathcal{M} \neq 0$, and therefore

$$\Gamma(X, \pi^*\mathcal{M}) = \Gamma\big(\mathbb{A}^{n+1}, j_*\pi^*\mathcal{M}\big) \neq 0.$$

It follows that there is some $\ell \in \mathbb{Z}$ such that $\Gamma_\ell(X, \pi^*\mathcal{M}) \neq 0$. On the other hand, we have $\Gamma_0(X, \pi^*\mathcal{M}) = 0$ by (22.6). We will show that this leads to a contradiction. Suppose first that $\ell \geq 1$. Take any nonzero element $s \in \Gamma_\ell(X, \pi^*\mathcal{M})$. Then

$$\theta s = \sum_{j=0}^{n} x_j \partial_j s = \ell s \neq 0,$$

and so at least one $\partial_j s \in \Gamma_{\ell-1}(X, \pi^*\mathcal{M})$ must be nonzero. Repeating this argument, we eventually arrive at $\Gamma_0(X, \pi^*\mathcal{M}) \neq 0$, which is a contradiction. The remaining possibility is that $\ell \leq -1$. Since $s \in \Gamma(X, \pi^*\mathcal{M})$ and $\pi^*\mathcal{M}$ is quasi-coherent, we cannot have $x_j s = 0$ for every j. It follows that $\Gamma_{\ell+1}(X, \pi^*\mathcal{M}) \neq 0$, and as before, this leads to a contradiction after finitely many steps. □

This result says, in particular, that coherent $\mathscr{D}_{\mathbb{P}^n}$-modules are the same thing as finitely generated modules over the ring of differential operators $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$. Let us briefly discuss the structure of this ring. We have

$$\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n}) \cong \Gamma_0\big(X, \mathscr{D}_{X \to \mathbb{P}^n}\big),$$

where $\mathscr{D}_{X \to \mathbb{P}^n} = \pi^*\mathscr{D}_{\mathbb{P}^n}$ is the transfer module. Recall from Lecture 16 that, in the case of a smooth morphism, $\mathscr{D}_{X \to \mathbb{P}^n}$ is the quotient of $\mathscr{D}_X$ by the submodule generated by the relative tangent bundle. In our setting, $\mathscr{D}_{X \to \mathbb{P}^n} \cong \mathscr{D}_X / \mathscr{D}_X \theta$, and so we recover the fact, already stated in Lecture 9, that $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$ consists of all differential operators on $\mathbb{A}^{n+1}$ that are homogenous of degree 0, modulo multiples of the Euler vector field θ.

One can turn this into a very concrete presentation by generators and relations, as follows. For $i, j \in \{0, 1, \dots, n\}$, set $D_{i,j} = x_i \partial_j$. A short calculation gives

$$[D_{i,j}, D_{k,\ell}] = \begin{cases} D_{i,i} - D_{j,j} & \text{if } k = j \text{ and } \ell = i, \\ D_{i,\ell} & \text{if } k = j \text{ and } \ell \neq i, \\ -D_{k,j} & \text{if } k \neq j \text{ and } \ell = i, \\ 0 & \text{if } k \neq j \text{ and } \ell \neq i. \end{cases} \tag{22.7}$$

We also have $\theta = D_{0,0} + D_{1,1} + \dots + D_{n,n}$. Then $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$ is generated as a non-commutative k-algebra by the $D_{i,j}$, and all the relations are generated by the above commutator relations and the additional relation $D_{0,0} + D_{1,1} + \dots + D_{n,n} = 0$.

Regular holonomic $\mathscr{D}$-modules of normal crossing type. We now turn to the classification of regular holonomic $\mathscr{D}$-modules of normal crossing type. Let me first explain what I mean by "normal crossing type". On $\mathbb{A}^n$, we can intersect the various components of the normal crossing divisor $x_1 \cdots x_n = 0$ to obtain a total of 2^n nonsingular closed subvarieties. (Here we use the convention that the empty intersection equals $\mathbb{A}^n$.) Their conormal bundles give us 2^n conical Lagrangian subvarieties of the cotangent bundle $T^*\mathbb{A}^n$. In the usual coordinate system $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ on the cotangent bundle, the union of all these Lagrangians is exactly the closed subset

$$Z(x_1\xi_1, \dots, x_n\xi_n);$$

indeed, on each component, we have either $x_j = 0$ or $\xi_j = 0$, for every $j = 1, \dots, n$. We say that a (necessarily holonomic) $\mathscr{D}_{\mathbb{A}^n}$-module $\mathcal{M}$ is *of normal crossing type* if its characteristic variety satisfies

$$\mathrm{Ch}(\mathcal{M}) \subseteq Z(x_1\xi_1, \dots, x_n\xi_n).$$

Example 22.8. On $\mathbb{A}^2$, the condition is that the characteristic variety has at most four irreducible components: the zero section, the conormal bundles to the two axes, and the cotangent space to the origin.

Here is a typical example, to get started.

Example 22.9. Consider the A_n-module $M = A_n/A_n(x_1\partial_1 - \alpha_1, \dots, x_n\partial_n - \alpha_n)$, where $\alpha_1, \dots, \alpha_n \in k$ are scalars. The characteristic variety is defined by the principal symbols of the n operators, hence is exactly the set $Z(x_1\xi_1, \dots, x_n\xi_n)$. In particular, M is holonomic; I will leave it as an exercise to check that M is regular in the sense of Kashiwara and Kawai.

The analogous definition on $\mathbb{P}^n$ has to include the hyperplane at infinity. In homogeneous coordinates $x_0, x_1, \dots, x_n$, we are therefore looking at the closed subset

$$Z(x_0\xi_0, x_1\xi_1, \dots, x_n\xi_n) \subseteq T^*\mathbb{P}^n;$$

note that even though the cotangent bundle is not trivial, the notation still makes sense because each $x_j\partial_j$ is a globally defined vector field on $\mathbb{P}^n$. We then say that a (necessarily holonomic) $\mathscr{D}_{\mathbb{P}^n}$-module $\mathcal{M}$ is *of normal crossing type* if

$$\mathrm{Ch}(\mathcal{M}) \subseteq Z(x_0\xi_0, x_1\xi_1, \dots, x_n\xi_n).$$

Our goal is to describe explicitly all regular holonomic $\mathscr{D}_{\mathbb{P}^n}$-modules of normal crossing type, at least when k is algebraically closed. It will help us that $\mathbb{P}^n$ is $\mathscr{D}$-affine. Our starting point is the following lemma.

Lemma 22.10. *Let $\mathcal{M}$ be a holonomic left $\mathscr{D}_{\mathbb{P}^n}$-module that is regular and of normal crossing type. Then there is a finite-dimensional k-vector space $V \subseteq \Gamma(\mathbb{P}^n, \mathcal{M})$ that generates $\Gamma(\mathbb{P}^n, \mathcal{M})$ as a $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$-module, and is preserved by $x_0\partial_0, \dots, x_n\partial_n$.*

Proof. Regularity means that there is a global good filtration $F_\bullet\mathcal{M}$ such that $\mathcal{I}_{\mathrm{Ch}(\mathcal{M})}$ annihilates $\mathrm{gr}^F\mathcal{M}$. Since $\mathrm{Ch}(\mathcal{M}) \subseteq Z(x_0\xi_0, x_1\xi_1, \dots, x_n\xi_n)$, this says concretely that we have

$$x_j\partial_j \cdot F_i\mathcal{M} \subseteq F_i\mathcal{M}$$

for every $j = 0, 1, \dots, n$ and $i \in \mathbb{Z}$. Since $F_i\mathcal{M}$ is a coherent $\mathscr{O}_{\mathbb{P}^n}$-module,

$$\Gamma(\mathbb{P}^n, F_i\mathcal{M}) \subseteq \Gamma(\mathbb{P}^n, \mathcal{M})$$

is a finite-dimensional k-vector space that is preserved by $x_0\partial_0, \dots, x_n\partial_n$. We showed during the proof of Corollary 22.4 that $\mathcal{M}$ is generated as a $\mathscr{D}_{\mathbb{P}^n}$-module by finitely many global sections. If we choose i large enough, these sections will be global sections of $F_i\mathcal{M}$, and so the subspace $V = \Gamma(\mathbb{P}^n, F_i\mathcal{M})$ actually generates $\Gamma(\mathbb{P}^n, \mathcal{M})$ as a module over $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$. □

Now $x_0\partial_0, \dots, x_n\partial_n$ are commuting endomorphisms of the finite-dimensional k-vector space V. Assuming that k is algebraically closed, we get a decomposition

$$V = \bigoplus_{\alpha \in k^{n+1}} V_\alpha$$

into generalized eigenspaces, where $V_\alpha \subseteq V$ consists of all vectors $v \in V$ such that $(x_j\partial_j - \alpha_j)^m v = 0$ for $j = 0, 1, \dots, n$ and $m \gg 0$. In other words, $x_j\partial_j - \alpha_j$ acts nilpotently on the subspace V_α. Of course, only finitely many of the V_α are actually nonzero; also note that we must have $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$, due to the fact that $\theta = x_0\partial_0 + \dots + x_n\partial_n$ acts trivially on V. If we define

$$A = \left\{ \alpha \in k^{n+1} \;\middle|\; \alpha_0 + \alpha_1 + \dots + \alpha_n = 0 \right\},$$

then the direct sum above is actually indexed by a finite subset of A. Since V generates $\Gamma(\mathbb{P}^n, \mathcal{M})$, we get a similar decomposition for the entire space of global sections.

Lemma 22.11. *Let $\mathcal{M}$ be a holonomic left $\mathscr{D}_{\mathbb{P}^n}$-module that is regular and of normal crossing type, and set $M = \Gamma(\mathbb{P}^n, \mathcal{M})$. We have a decomposition*

$$M = \bigoplus_{\alpha \in A} M_\alpha$$

into finite-dimensional k-vector spaces M_α, such that the operator $x_j \partial_j - \alpha_j$ acts nilpotently on M_α for $j = 0, 1, \dots, n$.

Proof. To be completely precise, we define, for every $\alpha \in A$, the subspace

$$M_\alpha = \left\{ s \in M \;\middle|\; (x_j \partial_j - \alpha_j)^m s = 0 \text{ for } j = 0, 1, \dots, n \text{ and } m \gg 0 \right\}.$$

Since different M_α are easily seen to be linearly independent, it suffices to prove that every $s \in M$ can be written as a sum of elements in finitely many M_α. This is true for elements of V by the discussion above; and for other elements, it follows from the fact that M is generated by V as a $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$-module. Indeed, $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$ is generated as a k-algebra by the operators $D_{i,j} = x_i \partial_j$, and since we already have the desired decomposition for elements of V, we only have to prove that

$$D_{i,j} \cdot M_\alpha \subseteq M_{\alpha + e_j - e_i},$$

where e_i is the i-th coordinate vector in k^{n+1}. But as $x_k \partial_k = D_{k,k}$, this follows immediately from the commutator relations

$$[D_{i,j}, D_{k,k}] = \begin{cases} 0 & \text{if } k = i = j, \\ D_{i,j} & \text{if } k = j \text{ and } k \neq i, \\ -D_{i,j} & \text{if } k = i \text{ and } k \neq j, \\ 0 & \text{if } k \neq i, j. \end{cases}$$

that we had proved earlier. □

Exercises.

Exercise 22.1. Verify the relations in (22.7), and prove that $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$ does have the claimed presentation by generators and relations.