Lecture 21 (April 15)

Derived equivalences and cohomology. Today, we go back to abelian varieties over the complex numbers. Let X and Y be two abelian varieties (of the same dimension g), and suppose that we have an integral transform

$$\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y).$$

We can use the kernel E on $X \times Y$ to construct an induced transformation

$$\Phi_E^H \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$$

in cohomology. Let $\mathrm{ch}(E) \in H^*(X \times Y, \mathbb{Q})$ be the Chern character of the complex $E \in \mathrm{D}^b(X \times Y)$, and let $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ be the two projections. Then define

$$\Phi_E^H \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}), \quad \Phi_E^H(\alpha) = (p_2)_*\bigl(p_1^*(\alpha) \cup \mathrm{ch}(E)\bigr),$$

where $(p_2)_*$ is the Gysin map in cohomology. Note that Φ_E^H does not respects degrees in general, because $\mathrm{ch}(E)$ can have components in many different degrees.

Example 21.1. When L is a line bundle on X, one has

$$\mathrm{ch}(L) = \exp c_1(L) = 1 + c_1(L) + \frac{1}{2!} c_1(L)^2 + \cdots$$

For a vector bundle E, the Chern character is a certain polynomial (with rational coefficients) in the Chern classes: using the splitting principle, if $L_1, \dots, L_r$ are the Chern roots of E, then

$$\mathrm{ch}(E) = \sum_{j=1}^r \mathrm{ch}(L_j).$$

For an arbitrary complex E, we can define the Chern character by choosing a bounded complex $\mathscr{E}^\bullet$ of locally free sheaves that is quasi-isomorphic to E, and then setting

$$\mathrm{ch}(E) = \sum_{j \in \mathbb{Z}} (-1)^j \, \mathrm{ch}(\mathscr{E}^j).$$

It can be shown that the alternating sum on the right-hand side is the same for every locally free resolution.

The construction is compatible with composition, in the sense that if

$$\begin{tikzcd} \mathrm{D}^b(X) \arrow[r, "\mathbf{R}\Phi_E"] \arrow[rr, bend left, "\mathbf{R}\Phi_{E*F}"] & \mathrm{D}^b(Y) \arrow[r, "\mathbf{R}\Phi_F"] & \mathrm{D}^b(Z) \end{tikzcd}$$

are two integral transforms (so that their composition is an integral transform with kernel $E * F$), then the induced diagram

$$\begin{tikzcd} H^*(X, \mathbb{Q}) \arrow[r, "\Phi_E^H"] \arrow[rr, bend left, "\Phi_{E*F}^H"] & H^*(Y, \mathbb{Q}) \arrow[r, "\Phi_F^H"] & H^*(Z, \mathbb{Q}) \end{tikzcd}$$

is also commutative. This is a consequence of the Grothendieck-Riemann-Roch theorem for Chern characters. Recall that the convolution is defined as

$$E * F = \mathbf{R}(p_{13})_*\bigl(p_{12}^* E \otimes p_{23}^* F\bigr).$$

As easy computation reduces the problem to showing that

$$\mathrm{ch}(E * F) = (p_{13})_*\bigl(p_{12}^* \,\mathrm{ch}(E) \cup p_{23}^* \,\mathrm{ch}(F)\bigr).$$

The Chern character always commutes with pulling back; and on abelian varieties, it also commutes with pushing forward. Indeed, if $f \colon X \to Y$ is a morphism between abelian varieties, and $E \in \mathrm{D}^b(X)$ a bounded complex of coherent sheaves, then the Grothendieck-Riemann-Roch theorem gives

$$\operatorname{ch}\bigl(\mathbf{R}f_*E\bigr) = \operatorname{td}(\mathscr{T}_Y) \cup \operatorname{ch}\bigl(\mathbf{R}f_*E\bigr) = f_*\bigl(\operatorname{td}(\mathscr{T}_X) \cup \operatorname{ch}(E)\bigr) = f_* \operatorname{ch}(E).$$

This works because the Todd class of the tangent bundle $\operatorname{td}(\mathscr{T}_X)$ is trivial on an abelian variety, due to the tangent bundle itself being trivial.

In particular, if $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is an equivalence, then the induced homomorphism $\Phi_E^H \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ is an isomorphism. Let me point out again that it is usually not compatible with the grading.

Example 21.2. Let's consider the Fourier transform $\mathbf{R}\Phi_P \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})$. To compute the induced homomorphism on cohomology, we first need to know the first Chern class $c_1(P)$ of the Poincaré bundle. From the Künneth formula, we get

$$H^2(X \times \hat{X}, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z}) \otimes H^1(\hat{X}, \mathbb{Z}) \oplus H^2(\hat{X}, \mathbb{Z}),$$

Now $c_1(P)$ belongs to the subspace $H^1(X, \mathbb{Z}) \otimes H^1(\hat{X}, \mathbb{Z})$, because the restriction of P to each slice $X \times \{\alpha\}$ and $\{x\} \times \hat{X}$ has trivial first Chern class (in cohomology). We can rewrite this subspace if we remember that $\hat{X} \cong \operatorname{Pic}^0(X) \cong H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z})$. This gives $H_1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z})$, and therefore

$$H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*.$$

Under this isomorphism, the first Chern class

$$c_1(P) \in H^1(X, \mathbb{Z}) \otimes H^1(X, \mathbb{Z})^*$$

is the identity: if $e_1, \dots, e_{2g} \in H^1(X, \mathbb{Z})$ is a basis, and $e_1^*, \dots, e_{2g}^* \in H^1(X, \mathbb{Z})^*$ the dual basis, then one can show that

$$c_1(P) = \sum_{j=1}^{2g} e_j \otimes e_j^*.$$

From this, it is easy to see that

$$\frac{1}{n!} c_1(P)^n \in H^n(X, \mathbb{Z}) \otimes H^n(X, \mathbb{Z})^*$$

has integer coefficients, and hence that the Chern character $\operatorname{ch}(P) \in H^*(X \times \hat{X}, \mathbb{Z})$ is also integral. It follows that

$$\Phi_P^H \colon H^*(X, \mathbb{Z}) \to H^*(\hat{X}, \mathbb{Z})$$

makes sense (and is an isomorphism) over the integers. Considering degrees, we get

$$\Phi_P^H \colon H^n(X, \mathbb{Z}) \to H^{2g-n}(\hat{X}, \mathbb{Z}).$$

One can prove (using the formula for the first Chern class of P) that this isomorphism is basically Poincaré duality: the product $(-1)^{n(n+1)/2+g} \cdot \Phi_P^H$ is the Poincaré duality isomorphism

$$H^n(X, \mathbb{Z}) \cong H_n(X, \mathbb{Z})^* \cong H^{2g-n}(X, \mathbb{Z})^* \cong H^{2g-n}(\hat{X}, \mathbb{Z}).$$

In fact, something similar happens for an arbitrary derived equivalence between two abelian varieties, as our the next proposition shows.

Proposition 21.3. *Let $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be an equivalence. Then*

$$\Phi_E^H \colon H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$$

is an isomorphism over the integers.

Proof. From $\mathbf{R}\Phi_E$, we constructed an induced equivalence

$$F_E\colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(Y \times \hat{Y}),$$

with the help of the diagram in (19.3). We also showed that F_E is tensor product with a certain line bundle $N_E \in \mathrm{Pic}(X \times \hat{X})$, followed by pushforward along the isomorphism $\varphi_E\colon X \times \hat{X} \to Y \times \hat{Y}$. Just as for the Poincaré bundle, the Chern character $\mathrm{ch}(N_E)$ is a class in the cohomology of $X \times \hat{X}$ with integer coefficients. Therefore the homomorphism in cohomology associated to F_E is an isomorphism $H^*(X \times \hat{X}, \mathbb{Z}) \cong H^*(Y \times \hat{Y}, \mathbb{Z})$. The two vertical arrows in (19.3) also induce isomorphisms on integral cohomology (because they involve only isomorphisms and the Poincaré bundle); therefore the homomorphism associated to the equivalence $\mathbf{R}\Phi_E \times \mathbf{R}\Phi_E^{-1}\colon \mathrm{D}^b(X \times X) \to \mathrm{D}^b(Y \times Y)$ is an isomorphism

$$H^*(X \times X, \mathbb{Z}) \cong H^*(Y \times Y, \mathbb{Z}).$$

A short computation shows that it acts as conjugation by Φ_E^H, and together with the Künneth decomposition, this is enough to conclude that

$$\Phi_E^H\colon H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$$

must be an isomorphism. □

Exercise 21.1. Let L be a line bundle on abelian variety X. Show that

$$\frac{1}{n!} c_1(L)^n \in H^{2n}(X, \mathbb{Z}),$$

and conclude that the Chern character $\mathrm{ch}(L)$ is an element of $H^*(X, \mathbb{Z})$.

Finally, we can give a cohomological criterion for when two abelian varieties X and Y are derived equivalent. The main point is that a complex abelian variety X can be reconstructed from the Hodge structure on $H^1(X, \mathbb{Z})$, meaning from the Hodge decomposition on

$$H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X).$$

Indeed, X is isomorphic to its own Albanese variety

$$\mathrm{Alb}(X) = H^0(X, \Omega_X^1)^* / H_1(X, \mathbb{Z}),$$

and we have $H_1(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$ and $H^0(X, \Omega_X^1) \cong H^{1,0}(X)$. So if X and Y are abelian varieties, and if $H^1(X, \mathbb{Z})$ and $H^1(Y, \mathbb{Z})$ are isomorphic as Hodge structures, then $X \cong Y$. (This is an isomorphism of compact complex manifolds, but since X and Y are projective, the isomorphism is automatically algebraic as well, due to Chow's theorem.)

Now let's go back to the criterion in Corollary 20.4, which says that $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ iff $U(X \times \hat{X}, Y \times \hat{Y}) \neq \emptyset$. This set of "unitary" isomorphisms was defined as follows. Write a given homomorphism $\varphi\colon X \times \hat{X} \to Y \times \hat{Y}$ in the form

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with $\alpha\colon X \to Y$, $\beta\colon \hat{X} \to Y$, $\gamma\colon X \to \hat{Y}$, and $\delta\colon \hat{X} \to \hat{Y}$; then take the dual homomorphisms $\hat{\alpha}\colon \hat{Y} \to \hat{X}$, $\hat{\beta}\colon \hat{Y} \to X$, $\hat{\gamma}\colon Y \to \hat{X}$, and $\hat{\delta}\colon Y \to X$, and assemble them into a second matrixp

$$\varphi^* = \begin{pmatrix} \hat{\delta} & -\hat{\beta} \\ -\hat{\gamma} & \hat{\alpha} \end{pmatrix}$$

that represents a homomorphism $\varphi^*\colon Y \times \hat{Y} \to X \times \hat{X}$. Then if $\varphi^* \circ \varphi = \mathrm{id}$, we say that $\varphi \in U(X \times \hat{X}, Y \times \hat{Y})$.

According to the discussion above, an isomorphism $\varphi\colon X \times \hat{X} \to Y \times \hat{Y}$ is the same thing as an isomorphism $f\colon H^1(X \times \hat{X}, \mathbb{Z}) \to H^1(Y \times \hat{Y}, \mathbb{Z})$ that respects the Hodge structures. The extra condition of being "unitary" can also be seen on cohomology. Writing

$$H^1(X \times \hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \oplus H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z})^*,$$

we have a natural bilinear pairing q_X, defined by the rule

$$q_X\big((\alpha_1, \phi_1), (\alpha_2, \phi_2)\big) = \phi_1(\alpha_2) + \phi_2(\alpha_1).$$

We can then restate the criterion from Corollary 20.4 as follows.

Corollary 21.4. *Let X and Y be abelian varieties. We have $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ if and only if there is an isomorphism of Hodge structures*

$$f \colon H^1(X \times \hat{X}, \mathbb{Z}) \to H^1(Y \times \hat{Y}, \mathbb{Z})$$

that is an isometry with respect to the bilinear pairings q_X and q_Y.

Proof. Choose a basis in $H^1(X, \mathbb{Z})$, and the dual basis in $H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$; then the pairing q_X is represented by the matrix

$$\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

We can represent the isomorphism $f \colon H^1(X \times \hat{X}, \mathbb{Z}) \to H^1(Y \times \hat{Y}, \mathbb{Z})$ as a matrix

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a \colon H^1(X, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$, $b \colon H^1(X, \mathbb{Z})^* \to H^1(Y, \mathbb{Z})$, and so on. The condition to be an isometry is then

$$\begin{pmatrix} c^* & a^* \\ d^* & d^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & b^* \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}.$$

Here $a^* \colon H^1(Y, \mathbb{Z})^* \to H^1(X, \mathbb{Z})^*$ is the homomorphism dual to a, and so on. Now $a \colon H^1(X, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$ is, by assumption, a morphism of Hodge structures, and so it corresponds to a morphism of abelian varieties $\alpha \colon X \to Y$. Under the isomorphisms $H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$ and $H^1(\hat{Y}, \mathbb{Z}) \cong H^1(Y, \mathbb{Z})^*$, the dual homomorphism a^* then corresponds exactly to the dual morphism $\hat{\alpha} \colon \hat{Y} \to \hat{X}$. Likewise, $b \colon H^1(X, \mathbb{Z})^* \to H^1(Y, \mathbb{Z})$ corresponds to a morphism $\beta \colon \hat{X} \to Y$, but the dual homomorphism $b^* \colon H^1(Y, \mathbb{Z}) \to H^1(X, \mathbb{Z})^{**} \cong H^1(X, \mathbb{Z})$ involves the isomorphism with the double dual, and for that reason, there is an extra sign: the corresponding morphism is $-\hat{\beta} \colon Y \to \hat{X}$. In this manner, the condition that f is an isometry turns into the identity

$$\begin{pmatrix} -\hat{\gamma} & \hat{\alpha} \\ \hat{\delta} & -\hat{\beta} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix},$$

which is exactly saying that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(X \times \hat{X}, Y \times \hat{Y}).$$

We now conclude by applying Corollary 20.4. ∎

Deligne's theorem on absolute Hodge classes. The next topic, which is going to take up the rest of the semester, is a theorem by Deligne about Hodge classes on abelian varieties. We started with a brief overview. Let X be a smooth projective variety over the complex numbers. For $k \geq 0$, we have the Hodge decomposition

$$H^{2k}(X, \mathbb{C}) = \bigoplus_{p+q=2k} H^{p,q}(X).$$

A cohomology class $\alpha \in H^{2k}(X, \mathbb{Z})$ is called an (integral) *Hodge class* if its image in $H^{2k}(X, \mathbb{C})$ lands in the subspace $H^{k,k}(X)$ – in other words, if it has type (k, k) with respect to the Hodge decomposition. Any closed subvariety $Z \subseteq X$ of codimension k has a fundamental class

$$[Z] \in H^{2k}(X, \mathbb{Z}),$$

and this is always a Hodge class. (*Proof:* Let $\mu \colon \tilde{Z} \to Z$ be a resolution of singularities; then $[Z]$ is Poincaré dual to the image of μ, and so

$$\int_{\tilde{Z}} \mu^* \alpha = \int_X [Z] \wedge \alpha$$

for every closed form $\alpha \in A^{2n-2k}(X)$, where $n = \dim X$. As $\dim \tilde{Z} = n - k$, the integral vanishes except when $\alpha \in A^{n-k,n-k}(X)$.) Hodge asked whether every integral Hodge class is "algebraic", meaning a linear combination of fundamental classes of subvarieties. Over $\mathbb{Z}$, there are counterexamples: cases where $[Z]$ is torsion (and therefore a Hodge class for trivial reasons), and cases where some multiple of $[Z]$ is algebraic, but $[Z]$ cannot be a linear combination of fundamental classes for degree reasons. The Hodge conjecture is therefore properly stated over $\mathbb{Q}$.

Conjecture 21.5 (Hodge)**.** *Every Hodge class in $H^{2k}(X, \mathbb{Q})$ is a $\mathbb{Q}$-linear combination of fundamental classes of subvarieties of codimension k.*

For $k = 1$, this is true even over $\mathbb{Z}$, by the Lefschetz $(1,1)$-theorem: every Hodge class in $H^2(X, \mathbb{Z})$ is the first Chern class of a line bundle. This works even on compact Kähler manifolds. But for larger values of k, the Hodge conjecture is known to be false on compact Kähler manifolds. (In fact, one can find compact complex tori that have Hodge classes in $H^{2k}(X, \mathbb{Q})$, but that don't contain any closed analytic subsets of codimension k.) So one has to use the fact that X is projective, and one way of doing this is by looking at arithmetic aspects (that make sense for polynomials but not for holomorphic functions). Deligne's theory of "absolute Hodge classes" is one step in this direction.

The general idea is as follows. Let $X \subseteq \mathbb{P}^N_{\mathbb{C}}$ be a smooth projective variety. It is of course the common zero set of finitely many homogeneous polynomials in $\mathbb{C}[z_0, \dots, z_N]$, and the finitely many coefficients of all these polynomials generate a subfield $k \subseteq \mathbb{C}$ that is finitely generated over $\mathbb{Q}$. By construction, X is defined over this much smaller field k. If we have an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we can apply it to the coefficients of the polynomials defining X, and obtain a new smooth projective variety X^σ. It is isomorphic to the original X as a variety over $\mathbb{Q}$ – if a point $[z_0, \dots, z_N]$ lies on X, then its image $[\sigma(z_0), \dots, \sigma(z_N)]$ lies on X^σ – but not over $\mathbb{C}$. In fact, not only are X and X^σ not isomorphic as complex manifolds, they are usually even not isomorphic as topological spaces.

The cohomology $H^*(X, \mathbb{C})$ can actually be computed algebraically (using the algebraic de Rham complex), and for that reason, $H^*(X, \mathbb{C}) \cong H^*(X^\sigma, \mathbb{C})$. The isomorphism is functorial, but it does not take the subspace $H^*(X, \mathbb{Q})$ to the subspace $H^*(X^\sigma, \mathbb{Q})$, because one needs the underlying topological space to define the cohomology with $\mathbb{Q}$-coefficients, and the underlying topological spaces of X and X^σ are not isomorphic. So if one has a Hodge class $\alpha \in H^{2k}(X, \mathbb{Q})$, there is no reason why its image $\alpha^\sigma \in H^{2k}(X^\sigma, \mathbb{C})$ should again be a Hodge class – it might not even be a rational cohomology class. On the other hand, the fundamental class $[Z]$ of a closed subvariety does remain a Hodge class, because the isomorphism $H^{2k}(X, \mathbb{C}) \cong H^{2k}(X^\sigma, \mathbb{C})$ takes $[Z]$ to $[Z^\sigma]$. This potentially different behavior between Hodge classes and algebraic classes motivates the following definition.

Definition 21.6. A Hodge class $\alpha \in H^{2k}(X, \mathbb{Q})$ is called *absolute* if for every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the image $\alpha^\sigma \in H^{2k}(X^\sigma, \mathbb{C})$ is again a Hodge class.

The Hodge conjecture then breaks up into two steps: (1) Show that every Hodge class is absolute. (2) Show that every absolute Hodge class is algebraic. Absolute Hodge classes don't make sense on compact Kähler manifolds, and so this limits the scope of the problem to smooth projective varieties.

On abelian varietes, the Hodge conjecture is still open (and while I am skeptical about the general case, I do think that the Hodge conjecture is true on abelian varieties). If you went to Markman's talk two weeks ago, you'll remember that he proved the Hodge conjecture for all 4-dimensional abelian varieties. The best general result that we have is the following cool theorem by Deligne.

Theorem 21.7 (Deligne)**.** *Every Hodge class on an abelian variety is absolute.*

In the rest of the semester, we'll talk about the proof of Deligne's theorem. It involves moduli spaces of abelian varieties; complex multiplication (CM) on abelian varieties; certain special Hodge classes called "Weil classess"; and other things.