Lecture 20: April 22

Fuchsian differential equations. Our next topic is regularity. Let me try to motivate the definition by talking about another classical topic, namely Fuchsian differential equations. We work over the complex numbers, and take X to be a small open disk containing the origin in $\mathbb{C}$. Consider a differential equation of the form $Pu = 0$, where

$$P = a_0(x)\partial^m + a_1(x)\partial^{m-1} + \cdots + a_m(x)$$

is a differential operator of order m with holomorphic coefficients $a_j(x)$. If $a_0(0) \neq 0$, then the equation has m linearly independent holomorphic solutions, determined by the initial conditions $u(0), u'(0), \dots, u^{(m-1)}(0)$. Another way to say this is that the $\mathscr{D}_X$-module $\mathscr{D}_X/\mathscr{D}_X P$ is isomorphic to $\mathscr{O}_X^{\oplus m}$, where the isomorphism takes a vector $(u_0, \dots, u_{m-1})$ to the image of $u_0 + u_1\partial + \cdots + u_{m-1}\partial^{m-1}$. Here $\mathscr{D}_X$ is the sheaf of linear differential operators with holomorphic coefficients.

If $a_0(0) = 0$, then the story becomes more complicated.

Example 20.1. Suppose that $P = x\partial - \alpha$ for some $\alpha \in \mathbb{C}$. Here the solution $u = x^\alpha = e^{\alpha \log x}$ is really only defined on sectors, because of the term $\log x$.

Example 20.2. Suppose that $P = x^2\partial - 1$. Here the solution $u = e^{-1/x}$ is single-valued, but has an essential singularity at the origin. This is bad.

We need some terminology to talk about the solutions to the equation $Pu = 0$. Let us denote by R the ring of holomorphic functions on X, and by K its fraction field; elements of K are meromorphic functions. Further, we use $\tilde{R}$ to denote the ring of multi-valued holomorphic functions on $X \setminus \{0\}$; by this we mean holomorphic functions on the universal covering space. Using the exponential function

$$\mathbb{C} \to \mathbb{C}^*, \quad t \mapsto e^{2\pi i t},$$

the universal covering space of a disk of radius r minus the origin is the half-plane $\operatorname{Im} t > \frac{1}{2\pi}\log(1/r)$. This means that $\tilde{R}$ is the ring of holomorphic functions on a suitable half-plane. For example, $\log x = 2\pi i t$ and $x^\alpha = e^{2\pi i \alpha t}$ belong to $\tilde{R}$.

We want to avoid essential singularities; this can be done by controlling the rate of growth of solutions near the origin. We say that a multi-valued holomorphic function $f \in \tilde{R}$ has *moderate growth* near the origin if on any sector

$$S = \left\{ x \in \mathbb{C} \;\middle|\; 0 < |x| < \varepsilon \text{ and } \theta_0 \leq \arg x \leq \theta_1 \right\},$$

there is an integer $k \geq 0$ and a constant $C \geq 0$ such that

$$|f(x)| \leq \frac{C}{|x|^k}$$

for every $x \in S$. Let $\tilde{R}^{mod} \subseteq \tilde{R}$ be the subring of multi-valued functions with moderate growth near the origin. The functions x^α and $(\log x)^\ell$ belong to $\tilde{R}^{mod}$ for every $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$.

Example 20.3. Suppose that f is a single-valued holomorphic function on the punctured disk $X \setminus \{0\}$. Then f has moderate growth near the origin iff f is meromorphic; the reason is that $x^k f$ extends to a holomorphic function on X by Riemann's extension theorem. Thus moderate growth prevents essential singularities.

Let me now remind you of the classical theorem by Fuchs. After shrinking X, if necessary, we can assume that the origin is the only zero of $a_0(x)$; we can then divide through by $a_0(x)$ to get a differential operator with meromorphic coefficients.

Theorem 20.4 (Fuchs). *Let $P = \partial^m + a_1(x)\partial^{m-1} + \cdots + a_m(x)$ be a differential operator of order m with $a_j(x) \in K$. The following two conditions are equivalent:*

(a) All multi-valued solutions $u \in \tilde{R}$ of the differential equation $Pu = 0$ have moderate growth near the origin.

(b) For $j = 1, \dots, n$, the function $a_j(x)$ has a pole of order at most j at the origin.

If the conditions in the theorem are satisfied, the differential equation $Pu = 0$ is said to be *regular at the origin.* There is another way to formulate the algebraic condition in (b). Using identity $x^j \partial^j = (x\partial)(x\partial - 1) \cdots (x\partial - j + 1)$, we get

$$x^m P = (x\partial)^m + b_1(x)(x\partial)^{m-1} + \cdots + b_m(x),$$

and (b) becomes the condition that $b_1(x), \cdots, b_m(x)$ are holomorphic functions.

Systems of differential equations and regularity. We will prove Theorem 20.4 by turning the problem into a system of first-order differential equations. If we set $u_1 = u, u_2 = \partial u, \dots, u_m = \partial^{m-1} u$, then $Pu = 0$ is of course equivalent to the system of m first-order equations

$$\begin{aligned} \partial u_1 &= u_2 \\ \partial u_2 &= u_3 \\ &\vdots \\ \partial u_{m-1} &= u_m \\ \partial u_m &= -(a_m u_1 + \cdots + a_1 u_m) \end{aligned}$$

More generally, let us consider a first-order system of the form

$$\partial u_i = \sum_{j=1}^m a_{i,j} u_j, \quad i = 1, \dots, m,$$

with m unknown functions $u_1, \dots, u_m$ and meromorphic coefficients $a_{i,j} \in K$. We can also write this in the form $\partial U = AU$, where U is the column vector with entries $u_1, \dots, u_m$, and A is an $m \times m$-matrix whose entries are meromorphic functions.

Example 20.5. If condition (b) is satisfied, we can instead look at the m functions $v_1 = u, v_2 = x\partial u, \dots, v_m = (x\partial)^{m-1} u$; the equation $Pu = 0$ is then also equivalent to the following system:

$$\begin{aligned} x\partial v_1 &= v_2 \\ x\partial v_2 &= v_3 \\ &\vdots \\ x\partial v_{m-1} &= v_m \\ x\partial v_m &= -(b_m v_1 + \cdots + b_1 v_m) \end{aligned}$$

In matrix notation, this becomes $x\partial V = BV$, where the entries of the $m \times m$-matrix B are now holomorphic functions.

Now let us describe the multi-valued solutions of such a system $\partial U = AU$. We can pull the system back to the universal covering space of $X \setminus \{0\}$, which amounts to setting $x = e^{2\pi i t}$. This gives us a system of first-order equations with holomorphic coefficients on a half-space; by Cauchy's theorem, it has m linearly independent holomorphic solutions $\tilde{u}^1, \dots, \tilde{u}^m$; here each $\tilde{u}^j$ is a column vector with entries in $\tilde{R}$. Let us denote by $\tilde{S}(t)$ the $m \times m$-matrix whose columns are $\tilde{u}^1, \dots, \tilde{u}^m$. Since the coefficients of the system are invariant under the substitution $t \mapsto t + 1$, the columns of $\tilde{S}(t+1)$ form another basis for the vector space of solutions, and so

$$\tilde{S}(t+1) = \tilde{S}(t)C$$

for a certain matrix $C \in \mathrm{GL}_n(\mathbb{C})$. This matrix is called the *monodromy matrix* of the system, because it describes how the multi-valued solutions to the system transform when going around the origin.

Choose a matrix Γ with the property that $C = e^{2\pi i \Gamma}$; such a matrix always exists, and is unique if we require that the eigenvalues of Γ have their real part in a fixed interval of unit lengt, such as $[0, 1)$. The matrix $\tilde{S}(t)e^{-2\pi i t \Gamma}$ is now invariant under the substitution $t \mapsto t + 1$, and so

$$\tilde{S}(t)e^{-2\pi i t \Gamma} = \Sigma(e^{2\pi i t}),$$

where $\Sigma(x)$ is an $m \times m$-matrix whose entries are holomorphic functions on $X \setminus \{0\}$. Replacing $2\pi i t$ by $\log x$, we see that the columns of the matrix

$$S(x) = \Sigma(x)e^{\log x \, \Gamma}$$

form a basis for the space of multi-valued solutions to the system $\partial U = AU$.

Changing the basis in the vector space of solutions amounts to conjugating C and Γ by the change-of-basis matrix. Since we are working over $\mathbb{C}$, we can therefore choose our basis in such a way that Γ is in Jordan canonical form. Thus Γ is block-diagonal, with blocks of the type

$$\begin{pmatrix} \alpha & 1 & & & \\ & \alpha & 1 & & \\ & & \ddots & \ddots & \\ & & & \alpha & 1 \\ & & & & \alpha \end{pmatrix}$$

which means that $e^{\log x \, \Gamma}$ is block-diagonal, with blocks of the type

$$x^\alpha \cdot \begin{pmatrix} 1 & L_1(x) & L_2(x) & \cdots & L_{m-1}(x) \\ & 1 & L_1(x) & \cdots & L_{m-2}(x) \\ & & \ddots & \ddots & \vdots \\ & & & 1 & L_1(x) \\ & & & & 1 \end{pmatrix}$$

where now $L_j(x) = \frac{1}{j!}(\log x)^j$. This gives a fairly concrete description of what multi-valued solutions look like.

Example 20.6. A corollary of the discussion so far is that any m-th order differential equation of the form $Pu = 0$ has a solution of the form $x^\alpha h(x)$, where $h(x)$ is holomorphic outside the origin, and $\alpha \in \mathbb{C}$ has the property that $e^{2\pi i \alpha}$ is an eigenvalue of the monodromy matrix C.

Now our goal is to prove a version of Theorem 20.4 for systems.

Definition 20.7. We say that two systems $\partial U = AU$ and $\partial V = BV$ are *equivalent* if there is a matrix $M(x) \in \mathrm{GL}_m(K)$ with meromorphic entries such that

$$B = \partial M \cdot M^{-1} + MAM^{-1}.$$

This means that U solves the first system iff $V = MU$ solves the second one.

Here is the analogue of Theorem 20.4 for systems.

Theorem 20.8. *Let A be an $m \times m$-matrix with entries in K. The following three conditions are equivalent:*

(a) All multi-valued solutions of $\partial U = AU$ have moderate growth near the origin, meaning that the individual components of U do.

(b) The system $\partial U = AU$ is equivalent to a system of the form $\partial V = x^{-1}\Gamma V$, where Γ is an $m \times m$-matrix with constant entries.

(c) *The system $\partial U = AU$ is equivalent to a system of the form $\partial V = x^{-1}BV$, where B is an $m \times m$-matrix with holomorphic entries.*

A system satisfying these equivalent conditions is called *regular* at the origin.

Proof. Let us show that (a) implies (b). We already know that a fundamental system of solutions is of the form $S(x) = \Sigma(x)e^{\log x\,\Gamma}$. By assumption, the entries of the matrix $S(x)$ have moderate growth near the origin. Since powers of $\log x$ have moderate growth, it follows that the entries of

$$\Sigma(x) = S(x)e^{-\log x\,\Gamma}$$

also have moderate growth near the origin. The entries of $\Sigma(x)$ are therefore meromorphic functions, and so $\Sigma(x) \in \mathrm{GL}_m(K)$. After replacing U by $V = \Sigma^{-1}(x)U$, we obtain the equivalent system

$$\partial V = \frac{1}{x}\Gamma V,$$

which is what we wanted to show.

It is clear that (b) implies (c), and so it remains to prove that (c) implies (a). Let V be any multi-valued solution of the system $x\partial V = BV$. Here V is a column vector with entries $v_1, \dots, v_m \in \tilde{R}$. To prove that $v_1, \dots, v_m \in \tilde{R}^{mod}$, we need to understand their asymptotic behavior on any sector

$$S = \left\{\, x \in \mathbb{C} \;\middle|\; 0 < |x| < \varepsilon \text{ and } \theta_0 \le \arg(x) \le \theta_1 \,\right\}.$$

Let us set $\|V\|^2 = |v_1|^2 + \dots + |v_m|^2$ and $x = re^{i\theta}$. Since the entries of B are holomorphic, they are bounded on S. A short calculation using $\partial V = x^{-1}BV$ gives

$$\frac{\partial}{\partial r}\|V\| \le \frac{1}{2\|V\|}\sum_{j=1}^{m} 2|v_j|\left|\frac{\partial v_j}{\partial r}\right| \le \sqrt{\sum_{j=1}^{m}\left|\frac{\partial v_j}{\partial x}\right|^2} \le \frac{C}{r}\|V\|,$$

where $C \ge 0$ is an upper bound for the matrix norm of B on the sector S. After integrating over r, this becomes

$$\|V(re^{i\theta})\| \le \|V(r_0e^{i\theta})\| + \int_r^{r_0} \frac{C}{s}\|V(se^{i\theta})\|\,ds,$$

for any $0 < r \le r_0 < \varepsilon$. Now we apply Grönwall's inequality to conclude that

$$\|V(re^{i\theta})\| \le \|V(r_0e^{i\theta})\| \exp\int_r^{r_0} \frac{C}{s}\,ds = \|V(r_0e^{i\theta})\| \left(\frac{r_0}{r}\right)^C.$$

This means exactly that all entries of V have moderate growth at the origin. □

Note. Grönwall's inequality says that an integral inequality of the form

$$f(t) \le C + \int_{t_0}^{t} g(s)f(s)\,ds$$

for a real function $f(t)$ implies that

$$f(t) \le C\exp\int_{t_0}^{t} g(s)\,ds.$$

We are now in a position to prove the theorem of Fuchs from the beginning.

Proof of Theorem 20.4. Consider a differential operator

$$P = \partial^m + a_1(x)\partial^{m-1} + \dots + a_m(x),$$

with $a_j \in K$. Suppose that each a_j has a pole of order at most j at the origin. As we remarked before, we can rewrite $x^mP = (x\partial)^m + b_1(x)(x\partial)^{m-1} + \dots + b_m(x)$, with $b_j \in R$ holomorphic. Setting $v_1 = u, v_2 = x\partial u, \dots, v_m = (x\partial)^{m-1}u$, it follows that the column vector $V = (v_1, \ldots, v_m)$ solves a system of the form $\partial V = x^{-1} BV$. By Theorem 20.8, the multi-valued functions $v_1, \ldots, v_n$ have moderate growth near the origin, and so in particular $u \in \tilde{R}^{mod}$.

Let us prove the converse. Suppose that all multi-valued solutions of $Pu = 0$ have moderate growth near the origin. If we write the corresponding system in the form $\partial U = AU$, then we have

$$t^m + a_1 t^{m-1} + \cdots + a_m = \det(t \,\mathrm{id} - A),$$

and so we can recover the coefficients of P from the characteristic polynomial of the matrix A. It is not hard to see that all solutions of $\partial U = AU$ also have moderate growth near the origin. By Theorem 20.8, our system is equivalent to a system of the form $\partial V = x^{-1} \Gamma V$, where Γ is an $m \times m$-matrix with constant entries. Consequently, there exists a matrix $M \in \mathrm{GL}_m(K)$ such that

$$A = \partial M \cdot M^{-1} + \frac{1}{x} M \Gamma M^{-1},$$

After clearing denominators, we get $M = x^\ell N$, with $N \in \mathrm{GL}_m(R)$. Then

$$A = \frac{1}{x}\Big(N \Gamma N^{-1} + \ell \,\mathrm{id}\Big) + \partial N \cdot N^{-1},$$

and if we now compute the characteristic polynomial, we find that the j-th coefficient a_j has a pole of order at most j at $x = 0$ (being equal to a sum of $j \times j$-minors of the matrix on the right-hand side). □

The theorem we have just proved has another interesting consequence.

Corollary 20.9. *Two regular systems are equivalent if and only if their monodromy matrices are conjugate.*

Proof. The proof of Theorem 20.8 shows that any regular system is equivalent to a system of the form

$$\partial U = \frac{1}{x} \Gamma U,$$

where Γ is an $m \times m$-matrix with constant entries, such that the monodromy matrix of the system is $e^{2\pi i \Gamma}$. If two such systems have conjugate monodromy matrices, then they are easily seen to be equivalent (via a constant matrix M.) To prove the converse, it is of course enough to consider systems of this special type. Suppose that two such systems with matrices Γ and Γ' are equivalent. This means that there exists a matrix $M \in \mathrm{GL}_m(K)$ such that

$$\frac{1}{x}\Gamma' = \partial M \cdot M^{-1} + \frac{1}{x} M \Gamma M^{-1}.$$

Write $M = x^\ell N$, with $N \in \mathrm{GL}_m(R)$. After clearing denominators, we get

$$\Gamma' = x \partial N \cdot N^{-1} + N(\Gamma + \ell \,\mathrm{id}) N^{-1},$$

and since Γ and Γ' are constant, we can now set $x = 0$ to obtain

$$\Gamma' = N(\Gamma + \ell \,\mathrm{id}) N^{-1}.$$

Since $e^{2\pi i \ell} = 1$, this implies that $e^{2\pi i \Gamma'} = N e^{2\pi i \Gamma} N^{-1}$. □

Exercises.

Exercise 20.1. Show directly that if two systems $\partial U = AU$ and $\partial V = BV$ are equivalent, then their monodromy matrices are conjugate to each other.