Lecture 19 (April 8)

Derived equivalences of abelian varieties. From Mukai's theorem, we know that an abelian variety X and its dual $\hat{X}$ have isomorphic derived categories. Let's say that two abelian varieties X and Y are *derived equivalent* if $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$. We would like to know exactly when this happens. This question was completely answered by Orlov and Polishchuk. The general idea is that $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ happens if and only if $X \times \hat{X} \cong Y \times \hat{Y}$ are isomorphic as abelian varieties (but only certain kinds of isomorphisms are allowed).

Let me first explain why the product $X \times \hat{X}$ shows up. This has to do with "automorphisms" of the derived category $\mathrm{D}^b(X)$, or more precisely auto-equivalences. A closed point $x \in X(k)$ defines an automorphism $t_x \colon X \to X$ by translation, and pullback along this automorphism is an auto-equivalence of the derived category:

$$t_x^* \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X).$$

Similarly, a closed point $\alpha \in \hat{X}(k)$ defines a line bundle $P_\alpha \in \mathrm{Pic}^0(X)$, and tensor product by P_α is also an auto-equivalence:

$$P_\alpha \otimes - \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X).$$

By composition, the closed points of $X \times \hat{X}$ therefore correspond to a family of auto-equivalences

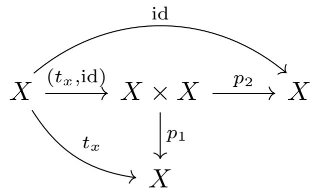
$$T_{(x,\alpha)} \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X), \quad T_{(x,\alpha)}(K) = P_\alpha \otimes t_x^* K \cong t_x^*(P_\alpha \otimes K).$$

Because X and $\hat{X}$ are varieties, this is a connected family; it contains $T_{(0,0)} = \mathrm{id}$. One can make sense of the group of auto-equivalences $\mathrm{Aut}\,\mathrm{D}^b(X)$ (using more fancy category theory); it has countably many connected components, and the neutral component (= the component containing the identity) is $X \times \hat{X}$. Now if $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$, then the automorphism groups of the two categories should be the same, and so $X \times \hat{X}$ should be isomorphic to $Y \times \hat{Y}$.

Orlov and Polishchuk make this heuristic argument precise, without actually defining the automorphism group $\mathrm{Aut}\,\mathrm{D}^b(X)$. It requires a careful study of the kernels of several different integral transforms. Each $T_{(x,\alpha)}$ is of course an integral transform: the kernel is the object

$$(t_x, \mathrm{id})_* P_\alpha \in \mathrm{D}^b(X \times X), \tag{19.1}$$

where the notation is as in the following diagram:



Indeed, with this choice, we get from the projection formula that

$$\mathbf{R}(p_2)_*\Big(p_1^* K \otimes (t_x, \mathrm{id})_* P_\alpha\Big) \cong \mathbf{R}(p_2)_*(t_x, \mathrm{id})_*\big(t_x^* K \otimes P_\alpha\big) \cong t_x^* K \otimes P_\alpha.$$

Now suppose that X and Y are two abelian varieties, whose derived categories $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ are equivalent. By Orlov's theorem, the equivalence is of the form

$$\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$$

for an object $E \in \mathrm{D}^b(X \times Y)$, unique up to isomorphism. We are going to associate to E an isomorphism of abelian varieties

$$\varphi_E \colon X \times \hat{X} \to Y \times \hat{Y},$$

by the following device. For each pair of closed points $(x, \alpha) \in X(k) \times \hat{X}(k)$, consider the auto-equivalence

$$T_{(x,\alpha)} \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$$

and its conjugate by $\mathbf{R}\Phi_E$, which is

$$\mathbf{R}\Phi_E \circ T_{(x,\alpha)} \circ \mathbf{R}\Phi_E^{-1} \colon \mathrm{D}^b(Y) \to \mathrm{D}^b(Y).$$

We'll argue below that this is again of the form $T_{\varphi_E(x,\alpha)}$ for a unique closed point $\varphi_E(x,\alpha) \in Y(k) \times \hat{Y}(k)$, starting from the fact that it is true for the closed point $(0,0)$, because $T_{(0,0)} = \mathrm{id}$.

The following lemma will be useful in describing the quasi-inverse $\mathbf{R}\Phi_E^{-1}$ as an integral transform. For a complex $E \in \mathrm{D}^b(X \times Y)$, we define

$$E^\vee = \mathbf{R}\mathcal{H}om_{\mathscr{O}_{X\times Y}}\big(E, \mathscr{O}_{X\times Y}\big).$$

Compare the following lemma with the formula for the inverse of the Fourier-Mukai transform.

Lemma 19.2. *Let $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be an equivalence of categories. Then the quasi-inverse is again an integral transform, with kernel*

$$E^\vee \otimes p_2^*\omega_Y[\dim Y].$$

Proof. The point is that the quasi-inverse $\mathbf{R}\Phi_E^{-1} \colon \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$ is necessarily the left-adjoint of $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$, because

$$\mathrm{Hom}_{\mathrm{D}^b(Y)}\big(A, \mathbf{R}\Phi_E(B)\big) \cong \mathrm{Hom}_{\mathrm{D}^b(X)}\big(\mathbf{R}\Phi_E^{-1}(A), B\big).$$

We can easily derive a formula for the left-adjoint:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}^b(Y)}\Big(A, \mathbf{R}\Phi_E(B)\Big) &\cong \mathrm{Hom}_{\mathrm{D}^b(Y)}\Big(A, \mathbf{R}(p_2)_*(E \overset{\mathbf{L}}{\otimes} p_1^*B)\Big) \\
&\cong \mathrm{Hom}_{\mathrm{D}^b(X)}\Big(p_2^*A, E \overset{\mathbf{L}}{\otimes} p_1^*B\Big) \\
&\cong \mathrm{Hom}_{\mathrm{D}^b(X)}\Big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee, p_1^*B\Big).
\end{aligned}$$

The exceptional inverse image functor (from Grothendieck duality) is

$$p_1^!B = p_1^*B \otimes p_2^*\omega_Y[\dim Y],$$

and by using Grothendieck duality, we can continue the calculation from above:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}^b(X)}\Big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee, p_1^*B\Big) &\cong \mathrm{Hom}_{\mathrm{D}^b(X)}\Big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee \otimes p_2^*\omega_Y[Y], p_1^!B\Big) \\
&\cong \mathrm{Hom}_{\mathrm{D}^b(X)}\Big(\mathbf{R}(p_1)_*\big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee \otimes p_2^*\omega_Y[Y]\big), B\Big)
\end{aligned}$$

This proves that

$$\mathbf{R}\Phi_E^{-1}(A) \cong \mathbf{R}(p_1)_*\big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee \otimes p_2^*\omega_Y[Y]\big)$$

is equivalent to an integral transform. □

Let's now return to our problem. Instead of trying to construct the isomorphism $\varphi_E \colon X \times \hat{X} \to Y \times \hat{Y}$ directly, we shall first define an equivalence

$$F_E \colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(Y \times \hat{Y}),$$

and then argue that F_E actually comes from an isomorphism between $X \times \hat{X}$ and $Y \times \hat{Y}$. This equivalence fits into the following commutative diagram:

$$\begin{array}{ccc} \mathrm{D}^b(X \times \hat{X}) & \xrightarrow{F_E} & \mathrm{D}^b(Y \times \hat{Y}) \\ \Big\downarrow \mathbf{R}\Phi_{A(X)} & & \Big\uparrow \mathbf{R}\Phi_{A(Y)}^{-1} \\ \mathrm{D}^b(X \times X) & \xrightarrow{\mathbf{R}\Phi_E \times \mathbf{R}\Phi_E^{-1}} & \mathrm{D}^b(Y \times Y) \end{array} \tag{19.3}$$

The vertical arrow is an equivalence

$$\mathbf{R}\Phi_{A(X)} \colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(X \times X)$$

that takes the skyscraper sheaf $k(x, \alpha)$ at a closed point $(x, \alpha) \in X(k) \times \hat{X}(\alpha)$ to the object in (19.1). Recall that this object is the kernel of the auto-equivalence $T_{(x,\alpha)} \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$. Think of this as saying that $X \times \hat{X}$ is the parameter space for all of these auto-equivalences. The correct kernel is

$$A(X) = \mu_* \big(p_{32}^* P_X \big) \in \mathrm{D}^b(X \times \hat{X} \times X \times X),$$

where the notation is as follows:

$$\begin{array}{ccc} X \times \hat{X} \times X & \xrightarrow{\mu} & X \times \hat{X} \times X \times X \\ \Big\downarrow p_{32} & & \\ X \times \hat{X} & & \end{array}$$

The two morphisms act on closed points as

$$\mu(x, \alpha, y) = (x, \alpha, x + y, y) \quad \text{and} \quad p_{32}(x, \alpha, y) = (y, \alpha).$$

With this choice, you can easily compute for yourself that

$$\mathbf{R}\Phi_{A(X)} \big(k(x, \alpha) \big) \cong (t_x, \mathrm{id})_* (P_\alpha).$$

The other vertical arrow in (19.3) is the quasi-inverse to $\mathbf{R}\Phi_{A(Y)}$; one can get an explicit formula for the kernel from Lemma 19.2.

Exercise 19.1. Verify that $\mathbf{R}\Phi_{A(X)}$ is indeed an equivalence. (*Hint:* Write it as the composition of an automorphism of $X \times X$ and Mukai's Fourier transform.)

The horizontal arrow in (19.3) is conjugation by $\mathbf{R}\Phi_E$. If we set $g = \dim Y$, then the kernel for $\mathbf{R}\Phi_E^{-1}$ is $E^\vee[g]$, and so the kernel representing conjugation is

$$p_{13}^* E^\vee[g] \overset{\mathbf{L}}{\otimes} p_{24}^* E \in \mathrm{D}^b(X \times X \times Y \times Y).$$

As the composition of three equivalences, $F_E \colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(Y \times \hat{Y})$ is an equivalence. It is also an integral transform for some $\tilde{E} \in \mathrm{D}^b(X \times \hat{X} \times Y \times \hat{Y})$. One can in principle derive a formula for the kernel $\tilde{E}$ (using convolutions), but the actual formula doesn't matter for us. Here is Orlov's theorem.

Theorem 19.4. *There is an isomorphism of abelian varieties*

$$\varphi_E \colon X \times \hat{X} \to Y \times \hat{Y}$$

and a line bundle $N_E \in \mathrm{Pic}(X \times \hat{X})$*, such that*

$$F_E = \mathbf{R}(\varphi_E)_* \big(N_E \otimes - \big).$$

Equivalently, the kernel representing F_E *is*

$$(\mathrm{id}, \varphi_E)_* N_E \in \mathrm{D}^b(X \times \hat{X} \times Y \times \hat{Y}).$$

We'll prove the theorem after looking at a few examples. A by-product of the construction is that for every pair of closed points $(x, \alpha) \in X(k) \times \hat{X}(k)$, the conjugated auto-equivalence

$$\mathbf{R}\Phi_E \circ T_{(x,\alpha)} \circ \mathbf{R}\Phi_E^{-1} \cong T_{\varphi_E(x,\alpha)}$$

is again of the same form. We can rewrite this identity as

$$\mathbf{R}\Phi_E \circ T_{(x,\alpha)} \cong T_{\varphi_E(x,\alpha)} \circ \mathbf{R}\Phi_E. \tag{19.5}$$

As another exercise, you can compute the convolutions of the two kernels on each side. The result is that if $\varphi_E(x, \alpha) = (y, \beta)$, then one has

$$(t_x \times \mathrm{id})_* \big(p_1^* P_{X,\alpha} \otimes E\big) \cong (\mathrm{id} \times t_y)^* E \otimes p_2^* P_{Y,\beta} \tag{19.6}$$

in $\mathrm{D}^b(X \times Y)$. In other words, the automorphism φ_E records how the kernel E responds to translations and tensor products on both X and Y.

Example 19.7. Consider the Fourier transform $\mathbf{R}\Phi_P \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})$. Here $Y = \hat{X}$ and $E = P$; by symmetry, $\hat{Y} \cong X$ and $\hat{P} = \sigma^* P$. What is the isomorphism

$$\varphi_P \colon X \times \hat{X} \to \hat{X} \times X$$

in this case? We can figure out the answer with very little pain if we make use of (19.6). Suppose that $\varphi_P(x, \alpha) = (\alpha', x')$. Then

$$(t_x \times \mathrm{id})_* \big(p_1^* P_\alpha \otimes P\big) \cong (\mathrm{id} \times t_{\alpha'})^* P \otimes p_2^* \hat{P}_{x'}$$

on $X \times \hat{X}$. By the seesaw theorem,

$$(\mathrm{id} \times t_{\alpha'})^* P \cong P \otimes p_1^* P_{\alpha'} \quad \text{and} \quad (t_x \times \mathrm{id})_* P \cong P \otimes p_2^* \hat{P}_{-x},$$

and so the identity from above becomes

$$p_1^* P_\alpha \otimes P \otimes p_2^* \hat{P}_{-x} \cong p_1^* P_{\alpha'} \otimes P \otimes p_2^* \hat{P}_{x'}.$$

Comparing the two sides, we find that $\alpha' = \alpha$ and $x' = -x$, and so

$$\varphi_P(x, \alpha) = (\alpha, -x).$$

This tells us how φ_P acts on closed points. Not surprisingly, one also has $N_P \cong P$ (but proving this takes a lot more work).

Here is another example where the line bundle N_E is nontrivial.

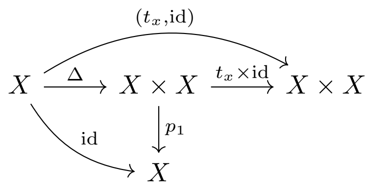
Example 19.8. Let $L \in \mathrm{Pic}(X)$, and consider $L \otimes - \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$. In this case, $Y = X$ and $E = \Delta_* L$. Let's again determine

$$\varphi_E \colon X \times \hat{X} \to X \times \hat{X}$$

with the help of (19.6). Suppose that $\varphi_E(x, \alpha) = (y, \beta)$. Then

$$(t_x \times \mathrm{id})_* \big(p_1^* P_\alpha \otimes \Delta_* L\big) \cong (\mathrm{id} \times t_y)^* \Delta_* L \otimes p_2^* P_\beta$$

We can simplify the left-hand side using the diagram



From the projection formula, we get

$$(t_x \times \mathrm{id})_* \big(p_1^* P_\alpha \otimes \Delta_* L\big) \cong (t_x \times \mathrm{id})_* \Delta_* (L \otimes P_\alpha) \cong (t_x, \mathrm{id})_* (L \otimes P_\alpha).$$

We can also simplify the right-hand side using the Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{(t_y,\mathrm{id})} & X \times X \\ \Big\downarrow{\scriptstyle t_y} & & \Big\downarrow{\scriptstyle \mathrm{id}\times t_y} \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

Flat base change (for the automorphism $\mathrm{id}\times t_y$ gives

$$(\mathrm{id}\times t_y)^*\Delta_* L \otimes p_2^* P_\beta \cong (t_y, \mathrm{id})_* t_y^* L \otimes p_2^* P_\beta.$$

If we compare the two sides of our original identity, we get

$$(t_x, \mathrm{id})_*(L \otimes P_\alpha) \cong (t_y, \mathrm{id})_* t_y^* L \otimes p_2^* P_\beta,$$

and therefore $y = x$ and $L \otimes P_\alpha \cong t_x^* L \otimes P_\beta$. When we looked at line bundles on abelian varieties, we defined the homomorphism

$$\phi_L \colon X \to \hat{X}, \quad P_{\phi_L(x)} \cong t_x^* L \otimes L^{-1}.$$

Substituting this into the above formula, we get $\alpha = \beta + \phi_L(x)$, and so

$$\varphi_E \colon X \times \hat{X} \to X \times \hat{X}, \quad \varphi_E(x, \alpha) = \bigl(x, \alpha - \phi_L(x)\bigr).$$

When $L \in \mathrm{Pic}^0(X)$ is translation invariant, φ_E is the identity; but otherwise, it isn't. One can also check that $N_E = p_1^* L$, and so the line bundle in Theorem 19.4 is nontrivial in this example.

Example 19.9. For the shift functor $[n] \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$, we have $Y = X$ and $E = \Delta_* \mathscr{O}_X[n]$. In this case, $E^\vee$ has a shift by $-n$ in it, and so the two cancel out; the result is that $\varphi_E = \mathrm{id}$ and $N_E = \mathscr{O}_{X\times\hat{X}}$. From the point of view of Theorem 19.4, a shift is therefore indistinguishable from the identity.

Proof of Orlov's theorem. Let's now prove Theorem 19.4. The equivalence

$$F_E \colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(Y \times \hat{Y})$$

from (19.3) is an integral transform with a certain kernel $\tilde{E} \in \mathrm{D}^b(X \times \hat{X} \times Y \times \hat{Y})$. It has two additional properties that we can make use of. The first is that

$$F_E\bigl(k(0,0)\bigr) \cong k(0,0).$$

Indeed, $\mathbf{R}\Phi_{A(X)}\bigl(k(0,0)\bigr)$ is the kernel corresponding to $T_{(0,0)}$, which is the identity. Conjugating by $\mathbf{R}\Phi_E$ takes this to

$$\mathbf{R}\Phi_E \circ T_{(0,0)} \circ \mathbf{R}\Phi_E^{-1} \cong T_{(0,0)},$$

because the identity of course commutes with $\mathbf{R}\Phi_E$. Under $\mathbf{R}\Phi_{A(Y)}^{-1}$, this goes back to the skyscraper sheaf $k(0,0)$ at the closed point $(0,0) \in Y(k) \otimes \hat{Y}(k)$.

The second property is that F_E is something like a homomorphism. Suppose that (x_1, α_2) and (x_2, α_2) are closed points such that

$$F_E\bigl(k(x_i, \alpha_i)\bigr) \cong k(y_i, \beta_i)$$

for closed points $(y_i, \beta_i) \in Y(k) \otimes \hat{Y}(k)$. This means that

$$\mathbf{R}\Phi_E \circ T_{(x_i,\alpha_i)} \circ \mathbf{R}\Phi_E^{-1} \cong T_{(y_i,\beta_i)}.$$

If we compose the two equivalences, we get

$$\begin{aligned} \mathbf{R}\Phi_E \circ T_{(x_1+x_2,\alpha_1+\alpha_2)} \circ \mathbf{R}\Phi_E^{-1} &\cong \mathbf{R}\Phi_E \circ T_{(x_1,\alpha_1)} \circ T_{(x_2,\alpha_2)} \circ \mathbf{R}\Phi_E^{-1} \\ &\cong \mathbf{R}\Phi_E \circ T_{(x_1,\alpha_1)} \circ \mathbf{R}\Phi_E^{-1} \circ \mathbf{R}\Phi_E \circ T_{(x_2,\alpha_2)} \circ \mathbf{R}\Phi_E^{-1} \\ &\cong T_{(y_1,\beta_1)} \circ T_{(y_2,\beta_2)} \cong T_{(y_1+y_2,\beta_1+\beta_2)}, \end{aligned}$$

because $t_{x_1} \circ t_{x_2} = t_{x_1+x_2}$ and $P_{\alpha_1} \otimes P_{\alpha_2} \cong P_{\alpha_1+\alpha_2}$. This is saying that the set

$$\left\{ (x,\alpha) \in X(k) \times \hat{X}(k) \;\middle|\; F_E\big(k(x,\alpha)\big) \cong k(y,\beta) \text{ for some } (y,\beta) \in Y(k) \times \hat{Y}(k) \right\}$$

is a *subgroup* of $X(k) \times \hat{X}(k)$. (In fact, we have shown that it contains the zero element and is closed under addition.)

Theorem 19.4 is therefore a consequence of the following abstract result about derived equivalences between abelian varieties. (The point is that the notation becomes much simpler if we consider arbitrary abelian varieties!)

Proposition 19.10. *Let X, Y be abelian varieties, and let $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be an equivalence. If the set*

$$\left\{ x \in X(k) \;\middle|\; \mathbf{R}\Phi_E\big(k(x)\big) \cong k(y) \text{ for some } y \in Y(k) \right\}$$

is a subgroup of $X(k)$, then $E \cong (\mathrm{id} \times \varphi)_* N$ for an isomorphism $\varphi \colon X \to Y$ and a line bundle $N \in \mathrm{Pic}(X)$.

Proof. For each closed point $x \in X(k)$, we set $E_x = E|_{\{x\} \times Y}$, so that

$$\mathbf{R}\Phi_P\big(k(x)\big) = E_x \in \mathrm{D}^b(Y).$$

As usual, we view these as a family of objects in the derived category $\mathrm{D}^b(Y)$, parametrized by the closed points of X. They form an algebraic family because $E \in \mathrm{D}^b(X \times Y)$ is a bounded complex of coherent sheaves on the product.

Let's first argue that E must be supported on the graph of a homomorphism $\varphi \colon X \to Y$. Let $S = \mathrm{Supp}\, E$ be the support of the complex E (= the union of the supports of all its cohomology sheaves). This is a closed subset of $X \times Y$. Consider the projection $p_1 \colon S \to X$. Because $E_0 \cong k(0)$, we know that $p_1^{-1}(0) = \{0\}$. By the theorem about fiber dimensions, the set of $x \in X(k)$ such that $\dim p_1^{-1}(x) = 0$ is the set of closed points of an open subscheme $U \subseteq X$; of course, $0 \in U(k)$. This means that E_x is supported on a finite set of points for $x \in U(k)$.

Because $\mathbf{R}\Phi_E$ is an equivalence, it is in particular fully faithful, and therefore

$$\mathrm{Hom}_{\mathrm{D}^b(Y)}(E_x, E_x) \cong \mathrm{Hom}_{\mathrm{D}^b(X)}\big(k(x), k(x)\big) \cong k.$$

If $\mathrm{Supp}\, E_x$ was two or more points, then E_x would split as a direct sum of complexes supported at each point, and then the left-hand side would have dimension ≥ 2. Similarly, if E_x had more than one nontrivial cohomology sheaf, we could again decompose E_x and get too many endomorphisms. Since $E_0 \cong k(0)$, it follows that for $x \in U(k)$, the complex E_x is actually a sheaf supported at a single closed point in $Y(k)$. If we denote this closed point by $\varphi(x) \in Y(k)$, then $\varphi \colon U \to Y$ is a morphism (because its graph is $S \cap U \times Y$). Now in fact

$$E_x \cong k\big(\varphi(x)\big);$$

indeed, you can easily check that if M is a finitely-generated module over a local k-algebra $(A, \mathfrak{m})$ such that $\mathrm{Supp}\, M = \{\mathfrak{m}\}$ and $\mathrm{Hom}_A(M, M) \cong k$, then $M \cong k$.

This says of course that $U(k)$ is contained in the subgroup

$$\left\{ x \in X(k) \;\middle|\; \mathbf{R}\Phi_E\big(k(x)\big) \cong k(y) \text{ for some } y \in Y(k) \right\}.$$

Because X is an abelian variety, any open neighborhood of 0 generates X as a group; therefore $U = X$, the morphism φ is defined on all of X, and $E_x \cong k\big(\varphi(x)\big)$ for every $x \in X(k)$. Since we also know that $\varphi(0) = 0$, we see that $\varphi \colon X \to Y$ is a homomorphism. It then follows from Nakayama's lemma that

$$E \cong (\mathrm{id}, \varphi)_* N$$

for a line bundle $N \in \mathrm{Pic}(X)$. It is a line bundle because its stalk at every point is a one-dimensional k-vector space. Therefore

$$\mathbf{R}\Phi_E \cong \mathbf{R}\varphi_*(N \otimes -),$$

and this can only be an equivalence if $\varphi\colon X \to Y$ is an isomorphism. $\square$