Lecture 18 (April 3)

Let X be an abelian variety, $\hat{X}$ the dual abelian variety, and P_X the Poincaré bundle on $X \times \hat{X}$. Last time, we introduced the symmetric Fourier-Mukai transform

$$\mathsf{FM}_X = \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})^{op},$$

which is defined as the composition of the (contravariant) duality functor

$$\mathbf{R}\Delta_X = \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(-, \omega_X[\dim X]\big) \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)^{op}$$

with Mukai's original Fourier transform

$$\mathbf{R}\Phi_P(K) = \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^* K \otimes P\big).$$

The main theorem is that the two contravariant functors

$$\mathsf{FM}_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})^{op} \quad \text{and} \quad \mathsf{FM}_{\hat{X}} \colon \mathrm{D}^b(\hat{X}) \to \mathrm{D}^b(X)^{op}$$

are mutually inverse equivalences of category.

Proof of Mukai's theorem. For clarity, let's denote the Poincaré bundle P_X on $X \times \hat{X}$ by the symbol P, and the Poincaré bundle $P_{\hat{X}}$ on $\hat{X} \times X$ by the symbol $\hat{P}$. The symmetric description of the dual abelian variety (in Lecture 14) shows that

$$\hat{P} \cong \sigma^* P$$

where $\sigma \colon X \times \hat{X} \to \hat{X} \times X$ swaps the two factors.

Now let's begin provin Mukai's theorem. Since we can interchange the role of X and $\hat{X}$, we only need to prove that the functor

$$\mathsf{FM}_{\hat{X}} \circ \mathsf{FM}_X = \mathbf{R}\Phi_{\hat{P}} \circ \mathbf{R}\Delta_{\hat{X}} \circ \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \tag{18.1}$$

is naturally isomorphic to the identity. We are going to use the standard derived category tools (such as flat base change and Grothendieck duality) to show that the composition is an integral transform (with a kernel on $X \times X$); and then we'll use properties of the Poincaré bundle to prove that the kernel is the structure sheaf of the diagonal (and hence that the composition is the identity).

Let's first consider the last three terms; they give us a covariant functor

$$\mathbf{R}\Delta_{\hat{X}} \circ \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X}).$$

A brief computation using Grothendieck duality shows that this functor is an integral transform, whose kernel is the complex

$$P^{-1} \otimes p_2^* \omega_{\hat{X}}[g] \tag{18.2}$$

on $X \times \hat{X}$; here $g = \dim X$. To see this, we take $K \in \mathrm{D}^b(X)$, and compute:

$$\begin{aligned} \mathbf{R}\Phi_P \mathbf{R}\Delta_X(K) &= \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^* \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(K, \omega_X[g]) \otimes P\big) \\ &\cong \mathbf{R}(p_2)_*\big(\mathbf{R}\mathcal{H}om_{\mathscr{O}_{X \times \hat{X}}}(\mathbf{L}p_1^* K, p_1^*\omega_X[g]) \otimes P\big). \end{aligned}$$

The local version of Grothendieck duality gives $\mathbf{R}\Delta_{\hat{X}} \circ \mathbf{R}(p_2)_* \cong \mathbf{R}(p_2)_* \mathbf{R}\Delta_{X \times \hat{X}}$. If we apply this to the result of the preceding computation, we get

$$\begin{aligned} \mathbf{R}\Delta_{\hat{X}} \mathbf{R}\Phi_P \mathbf{R}\Delta_X(K) &\cong \mathbf{R}(p_2)_* \mathbf{R}\Delta_{X \times \hat{X}}\Big(\mathbf{R}\mathcal{H}om_{\mathscr{O}_{X \times \hat{X}}}(\mathbf{L}p_1^* K, p_1^*\omega_X[g]) \otimes P\Big) \\ &\cong \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^* K \otimes p_2^*\omega_{\hat{X}}[g] \otimes P^{-1}\big), \end{aligned}$$

because $\omega_{X \times \hat{X}} \cong p_1^*\omega_X \otimes p_2^*\omega_{\hat{X}}$, and because the two $\mathbf{R}\mathcal{H}om$'s cancel each other. This is an integral transform with kernel (18.2).

Now we need to compose this with $\mathbf{R}\Phi_{\hat{P}}$. By the computation from last time, the composition is again an integral transform; the kernel is the convolution of

(18.2) on $X \times \hat{X}$ with the line bundle $\hat{P}$ on $\hat{X} \times X$. Thus (18.1) is also an integral transform, with kernel

$$\mathbf{R}(p_{13})_*\Big(p_{12}^*P^{-1} \otimes p_2^*\omega_{\hat{X}}[g] \otimes p_{23}^*\hat{P}\Big).$$

In order to avoid ridiculous notation in the computation below, we now swap the second and third factor in $X \times \hat{X} \times X$. Since $\hat{P} \cong \sigma^* P$, we can then rewrite the kernel for (18.1) as

$$\mathbf{R}(p_{12})_*\Big(p_3^*\omega_{\hat{X}}[g] \otimes p_{13}^*P^{-1} \otimes p_{23}^*P\Big) \tag{18.3}$$

on $X \times X$. Theorem 17.12 will be proved once we show that (18.3) is isomorphic to the structure sheaf of the diagonal in $X \times X$.

Let $s\colon X \times X \to X$ be defined as $s = m \circ (i \times \mathrm{id})$; the formula on closed points is $s(x, y) = y - x$. The theorem of the cube shows that

$$p_{13}^*P^{-1} \otimes p_{23}^*P \cong (s \times \mathrm{id})^*P.$$

We can now apply flat base change in the commutative diagram

$$\begin{tikzcd} X \times X \times \hat{X} \arrow[r, "p_{12}"] \arrow[d, "s \times \mathrm{id}"] & X \times X \arrow[d, "s"] \\ X \times \hat{X} \arrow[r, "p_1"] & X \end{tikzcd}$$

and conclude that the complex in (18.3) is isomorphic to

$$\mathbf{L}s^*\, \mathbf{R}(p_1)_*\Big(P \otimes p_2^*\omega_{\hat{X}}[g]\Big) = \mathbf{L}s^*\, \mathsf{FM}_{\hat{X}}(\mathscr{O}_{\hat{X}}) = \mathbf{L}s^* k(0) = \Delta_* \mathscr{O}_X,$$

where $\Delta\colon X \to X \times X$ is the diagonal embedding. Here we used the fact that the symmetric Fourier-Mukai transform of the structure sheaf $\mathscr{O}_{\hat{X}}$ is the structure sheaf $k(0)$ of the closed point $0 \in X(k)$, as in (17.13). Because the integral transform with kernel $\Delta_*\mathscr{O}_X$ is the identity, this concludes the proof of Theorem 17.12.

Properties of the Fourier-Mukai transform. If we wanted to summarize the above proof in one line, it would be that the Fourier-Mukai transform is an equivalence because of the identity $p_{13}^*P^{-1} \otimes p_{23}^*P \cong (s \times \mathrm{id})^*P$ for the Poincaré bundle. The other formulas involving the Poincaré bundle that we have proved also lead to interesting properties of FM_X.

The first topic is how the Fourier-Mukai transform interacts with pulling back or pushing forward by a homomorphism between abelian varieties. Mukai only looked at the case of isogenies; the general case is due to Chen and Jiang.

Proposition 18.4. *Let $f\colon X \to Y$ be a homomorphism of abelian varieties over k. Then one has natural isomorphisms of functors*

$$\mathsf{FM}_Y \circ \mathbf{R}f_* \cong \mathbf{L}\hat{f}^* \circ \mathsf{FM}_X \quad \textit{and} \quad \mathsf{FM}_X \circ \mathbf{L}f^* \cong \mathbf{R}\hat{f}_* \circ \mathsf{FM}_Y,$$

where $\hat{f}\colon \hat{Y} \to \hat{X}$ is the induced homomorphism between the dual abelian varieties.

Proof. It will be enough to show that

$$\mathsf{FM}_Y \circ \mathbf{R}f_* \cong \mathbf{L}\hat{f}^* \circ \mathsf{FM}_X;$$

the second identity in the theorem follows from this with the help of Theorem 17.12. Using the definition of FM_Y and Grothendieck duality, we obtain

$$\mathsf{FM}_Y \circ \mathbf{R}f_* \cong \mathbf{R}\Phi_{P_X} \circ \mathbf{R}\Delta_Y \circ \mathbf{R}f_* \cong \mathbf{R}\Phi_{P_Y} \circ \mathbf{R}f_* \circ \mathbf{R}\Delta_X.$$

This reduces the problem to proving that

$$\mathbf{R}\Phi_{P_Y} \circ \mathbf{R}f_* \cong \mathbf{L}\hat{f}^* \circ \mathbf{R}\Phi_{P_X}. \tag{18.5}$$

We make use of the following commutative diagram:

$$\begin{tikzcd}
\hat{X} & \hat{Y} \arrow[l, "\hat{f}"'] & & \\
X \times \hat{X} \arrow[u, "p_2"] & X \times \hat{Y} \arrow[l, "\mathrm{id} \times \hat{f}"'] \arrow[u, "p_2"] \arrow[r, "f \times \mathrm{id}"] \arrow[d, "p_1"] & Y \times \hat{Y} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{Y} \\
& X \arrow[r, "f"] & Y &
\end{tikzcd}$$

The identity in (14.2), which followed from the universal property of the dual abelian variety, gives us $(f \times \mathrm{id})^* P_Y \cong (\mathrm{id} \times \hat{f})^* P_X$. Using the projection formula and flat base change, we can write the following chain of isomorphisms:

$$\begin{aligned}
\mathbf{R}\Phi_{P_Y} \circ \mathbf{R}f_* &\cong \mathbf{R}(p_2)_*\big(P_Y \otimes p_1^* \mathbf{R}f_*\big) \cong \mathbf{R}(p_2)_*\big(P_Y \otimes \mathbf{R}(f \times \mathrm{id})_* p_1^*\big) \\
&\cong \mathbf{R}(p_2)_*\big((f \times \mathrm{id})^* P_Y \otimes p_1^*\big) \cong \mathbf{R}(p_2)_*\big((\mathrm{id} \times \hat{f})^* P_X \otimes p_1^*\big) \\
&\cong \mathbf{R}(p_2)_* \mathbf{L}(\mathrm{id} \times \hat{f})^* \big(P_X \otimes p_1^*\big) \cong \mathbf{L}\hat{f}^* \, \mathbf{R}(p_2)_*\big(P_X \otimes p_1^*\big) \\
&\cong \mathbf{L}\hat{f}^* \circ \mathbf{R}\Phi_{P_X}
\end{aligned}$$

This calculation establishes Proposition 18.4. □

The symmetric Fourier-Mukai transform also exchanges translations and tensoring by the corresponding line bundles. Any closed point $x \in X(k)$ determines a translation morphism $t_x \colon X \to X$; on closed points, it is given by the formula $t_x(y) = x+y$. Since $X(k) \cong \mathrm{Pic}^0(\hat{X})$, it also determines a line bundle $\hat{P}_x \in \mathrm{Pic}^0(\hat{X})$.

Proposition 18.6. *Let $x \in X(k)$ and $\alpha \in \hat{X}(k)$ be closed points. Then one has natural isomorphisms of functors*

$$\mathsf{FM}_X \circ (t_X)_* = (\hat{P}_X \otimes -) \circ \mathsf{FM}_X \quad \textit{and} \quad \mathsf{FM}_X \circ (P_\alpha \otimes -) = (t_\alpha)_* \circ \mathsf{FM}_X,$$

where $\hat{P}_x \in \mathrm{Pic}^0(X)$, and $P_\alpha \in \mathrm{Pic}^0(\hat{X})$, are the corresponding line bundles.

Together with (17.13), this leads to the pleasant formulas

$$\mathsf{FM}_X\big(k(x)\big) = \hat{P}_x \quad \text{and} \quad \mathsf{FM}_X(P_\alpha) = k(\alpha),$$

for any pair of closed points $x \in X(k)$ and $\alpha \in \hat{X}(k)$. This symmetry is another reason for the name "symmetric" Fourier-Mukai transform.

Proof. Once again, it suffices to prove that

$$\mathsf{FM}_X \circ (t_x)_* = (\hat{P}_x \otimes -) \circ \mathsf{FM}_X$$

because the other identity follows from this with the help of Theorem 17.12. Using Grothendieck duality, we get a natural isomorphism of functors

$$\mathsf{FM}_X \circ (t_x)_* = \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \circ (t_x)_* = \mathbf{R}\Phi_P \circ (t_x)_* \circ \mathbf{R}\Delta_X,$$

and so the problem is reduced to showing that

$$\mathbf{R}\Phi_P \circ (t_x)_* = (\hat{P}_x \otimes -) \circ \mathbf{R}\Phi_P.$$

We use the following commutative diagram:

$$\begin{tikzcd}
X \times \hat{X} \arrow[r, "t_x \times \mathrm{id}"] \arrow[d, "p_1"] & X \times \hat{X} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{X} \\
X \arrow[r, "t_x"] & X &
\end{tikzcd}$$

Since $(t_x \times \mathrm{id})^* P \cong p_2^* \hat{P}_x \otimes P$ (by the seesaw theorem), we have

$$\begin{aligned}\mathbf{R}\Phi_P \circ (t_x)_* &= \mathbf{R}(p_2)_*\Big(P \otimes p_1^*(t_x)_*\Big) = \mathbf{R}(p_2)_*\Big(P \otimes (t_x \times \mathrm{id})_* p_1^*\Big) \\ &= \mathbf{R}(p_2)_*\Big((t_x \times \mathrm{id})^* P \otimes p_1^*\Big) = \mathbf{R}(p_2)_*\Big(p_2^* \hat{P}_x \otimes P \otimes p_1^*\Big) \\ &= \hat{P}_x \otimes \mathbf{R}(p_2)_*\big(P \otimes p_1^*\big) = \hat{P}_x \otimes \mathbf{R}\Phi_P,\end{aligned}$$

which is exactly what we need. □

The third property is more of an extended example. Let L be an ample line bundle on the abelian variety X. Mukai's Fourier transform

$$\mathbf{R}\Phi_P(L) = \mathbf{R}(p_2)_*\big(p_1^* L \otimes P\big)$$

is a vector bundle of rank $\dim H^0(X, L)$ on the dual abelian variety $\hat{X}$. The reason is that on each fiber of $p_2 \colon X \times \hat{X} \to \hat{X}$, the line bundle $L \otimes P_\alpha$ is again ample, and so all of its higher cohomology groups vanish; we know this at least over $\mathbb{C}$, where it follows from the Kodaira vanishing theorem. By cohomology and base change, we therefore have $R^i(p_2)_*\big(p_1^* L \otimes P\big) = 0$ for $i \neq 0$; for $i = 0$, we get a locally free sheaf $\mathscr{E}_L$ of rank $\dim H^0(X, L)$.

To see what $\mathscr{E}_L$ actually looks like, let's pull it back by the isogeny

$$\varphi_L \colon X \to \hat{X};$$

recall that this has the property that $t_x^* L \cong L \otimes P_{\phi_L(x)}$ for all closed points $x \in X(k)$. The key identity (which we used for the construction of the Poincaré bundle) is that

$$(\mathrm{id} \times \varphi_L)^* P \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}.$$

Now let's do the computation, using the following commutative diagram:

$$\begin{array}{ccc} X \times X & \xrightarrow{p_2} & X \\ \downarrow{\scriptstyle \mathrm{id} \times \varphi_L} & & \downarrow{\scriptstyle \varphi_L} \\ X \times \hat{X} & \xrightarrow{p_2} & \hat{X} \\ \downarrow{\scriptstyle p_1} & & \\ X & & \end{array}$$

Applying flat base change and the projection formula, we get

$$\begin{aligned}\varphi_L^* \mathscr{E}_L = \mathbf{L}\varphi_L^* \mathbf{R}(p_2)_*\big(P \otimes p_1^* L\big) &\cong \mathbf{R}(p_2)_*\big((\mathrm{id} \times \varphi_L)^* P \otimes p_1^* L\big) \\ &\cong \mathbf{R}(p_2)_*\big(m^* L \otimes p_2^* L^{-1}\big) \cong L^{-1} \otimes \mathbf{R}(p_2)_* m^* L.\end{aligned}$$

Now we need a small trick. We can write $m \colon X \times X \to X$ as a composition

$$X \times X \xrightarrow{f} X \times X \xrightarrow{p_1} X,$$

where $f \colon X \times X \to X \times X$ is the automorphism $f(x, y) = (x + y, y)$. Therefore

$$\mathbf{R}(p_2)_* m^* L \cong \mathbf{R}(p_2)_* \mathbf{R} f_*(f^* p_1^* L) \cong \mathbf{R}(p_2)_* p_1^* L \cong H^0(X, L) \otimes \mathscr{O}_X,$$

where the second step is the projection formula, and the third flat base change. So

$$\varphi_L^* \mathscr{E}_L \cong H^0(X, L) \otimes L^{-1}. \tag{18.7}$$

Note that L was ample, but that Mukai's Fourier transform $\mathbf{R}\Phi_P$ takes it to the *dual* of an ample vector bundle.

Here is the result for the symmetric Fourier-Mukai transform; this is better, because positivity is preserved.

Proposition 18.8. *Let L be an ample line bundle on X. Then $\mathsf{FM}_X(L)$ is an ample vector bundle of rank* $\dim H^0(X,L)$*, and one has*

$$\varphi_L^* \mathsf{FM}_X(L) \cong i^* L \otimes H^0(X,L)^*,$$

where $i\colon X \to X$ is the inversion morphism.

This follows directly from (18.7), together with the following formula (that gives an alternative description of the symmetric Fourier-Mukai transform):

$$\mathsf{FM}_X(K) \cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(i^* \mathbf{R}\Phi_P(K), \mathscr{O}_{\hat{X}}\big) \tag{18.9}$$

To prove it, we take an object $K \in \mathrm{D}^b(X)$ and start computing:

$$\mathsf{FM}_X(K) = \mathbf{R}(p_2)_* \Big(\mathbf{L}p_1^* \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(K, \omega_X[g]\big) \otimes P\Big)$$

We would like to interchange $\mathbf{R}(p_2)_*$ and $\mathbf{R}\mathcal{H}om$, and for that, we need to move all the terms on the right-hand side into the first argument of $\mathbf{R}\mathcal{H}om$. Here it helps that $\omega_{X\times\hat{X}} \cong p_1^*\omega_X \otimes p_2^*\omega_{\hat{X}}$ and that $P \cong (i\times \mathrm{id})^* P^{-1}$. Accordingly,

$$\begin{aligned}
\mathbf{L}p_1^* \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(K, \omega_X[g]\big) \otimes P &\cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_{X\times\hat{X}}}\Big(\mathbf{L}p_1^* K, p_1^*\omega_X[g]\Big) \otimes (i\times\mathrm{id})^* P^{-1}\Big) \\
&\cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_{X\times\hat{X}}}\Big(\mathbf{L}p_1^* K \otimes (i\times\mathrm{id})^* P \otimes p_2^*\omega_{\hat{X}}[g], \omega_{X\times\hat{X}}[2g]\Big) \\
&= \mathbf{R}\Delta_{X\times\hat{X}}\Big(\mathbf{L}p_1^* K \otimes (i\times\mathrm{id})^* P \otimes p_2^*\omega_{\hat{X}}[g]\Big)
\end{aligned}$$

If we put this into the formula from above and use the relative version of Grothendieck duality, we obtain

$$\begin{aligned}
\mathsf{FM}_X(K) &\cong \mathbf{R}\Delta_{\hat{X}} \mathbf{R}(p_2)_* \Big(\mathbf{L}p_1^* K \otimes (i\times\mathrm{id})^* P \otimes p_2^*\omega_{\hat{X}}[g]\Big) \\
&\cong \mathbf{R}\Delta_{\hat{X}}\Big(\omega_{\hat{X}}[g] \otimes \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^* K \otimes (i\times\mathrm{id})^* P\big)\Big) \\
&\cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_{\hat{X}}}\Big(\mathbf{R}(p_2)_*\big(\mathbf{L}p_1^* K \otimes (i\times\mathrm{id})^* P\big), \mathscr{O}_{\hat{X}}\Big) \\
&\cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_{\hat{X}}}\Big(\mathbf{L}i^* \mathbf{R}\Phi_P(K), \mathscr{O}_{\hat{X}}\Big).
\end{aligned}$$

The last step follows from the projection formula. So (18.9) is proved.

Note that L has rank 1 and $h^0(X,L)$ many global sections, wherease $\mathsf{FM}_X(L)$ has rank $h^0(X,L)$ and one global section (by Proposition 18.4). So the Fourier-Mukai transform takes ample line bundles to ample vector bundles, but interchanges "rank" and "dimension of the space of global sections". More generally, FM_X tends to interchange "local" and "global" data. This can be very useful in geometric applications of the Fourier-Mukai transform (such as "generic vanishing theory"). The reason is that there are two sets of tools: local tools (such as commutative algebra in regular local rings) and global tools (such as vanishing theorems), and a local (or global) problem on X may become tractable once we convert it into a global (or local) problem on $\hat{X}$.