Lecture 16 (March 27)

The derived category. Derived categories were introduced to have a better foundation for the theory of derived functors. When we calculate derived functors such as Tor or Ext, we typically find a (locally free, or flat, or injective) resolution of our given module/sheaf, apply the functor in question to each term of the resolution, and then take cohomology. The main idea behind the derived category is to keep not just the cohomology modules/sheaves, but the complexes themselves. Because the same module/sheaf can be resolved in many different ways, keeping the complex only makes sense if we declare different complexes obtained in this way to be isomorphic. This leads to the notion of a *quasi-isomorphism*: a morphism between two complexes that induces isomorphisms on cohomology.

Example 16.1. Consider the case of modules over a ring. Every module M has a (typically infinite) free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

and in the derived category, we want to consider the complex $F_\bullet$ as being isomorphic to M. If $G_\bullet$ is another free resolution of M, then a basic result in homological algebra says that there is a morphism of complexes $f\colon F_\bullet \to G_\bullet$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d"] \arrow[d, "f"] & F_1 \arrow[r, "d"] \arrow[d, "f"] & F_0 \arrow[r] \arrow[d, "f"] & M \arrow[d, "\mathrm{id}"] \\
\cdots \arrow[r] & G_2 \arrow[r, "d"] & G_1 \arrow[r, "d"] & G_0 \arrow[r] & M
\end{tikzcd}$$

commute. This morphism is only unique up to homotopy: for any other choice $f'\colon F_\bullet \to G_\bullet$, there are homomorphisms $s\colon F_n \to G_{n+1}$ such that $f' - f = ds + sd$.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dashed] & F_1 \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dashed, "s"'] & F_0 \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "s"'] & M \arrow[d, "\mathrm{id}"] \\
\cdots \arrow[r] & G_2 \arrow[r, "d"] & G_1 \arrow[r, "d"] & G_0 \arrow[r] & M
\end{tikzcd}$$

If we want to consider M, $F_\bullet$, and $G_\bullet$ as being isomorphic to each other, the two liftings of $\mathrm{id}\colon M \to M$ should be equal, and so we are forced to consider morphisms of complexes up to homotopy.

Example 16.2. In other cases, say for computing Ext, we might want to replace M by an injective resolution of the form

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots,$$

Now an injective resolution and a free resolution do not have much in common; the only thing we can say is that we have a morphism of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d"] \arrow[d] & F_1 \arrow[r, "d"] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & I^0 \arrow[r, "d"] & I^1 \arrow[r, "d"] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

that is an isomorphism on the level of cohomology—being resolutions of M, both complexes have cohomology only in degree zero. If we want both complexes to be isomorphic as objects of the derived category, we need to make sure that such quasi-isomorphisms have inverses.

Quasi-isomorphisms also arise naturally if we consider resolutions of complexes.

Example 16.3. An injective resolution of a complex $M^\bullet$ of modules is a complex $I^\bullet$ of injective modules, and a morphism of complexes $M^\bullet \to I^\bullet$ that induces isomorphisms on cohomology. This generalizes the usual definition for a single module to complexes.

Unfortunately, not every quasi-isomorphism has an inverse. The following example (in the category of $\mathbb{Z}$-modules) shows one way in which this can happen.

Example 16.4. In the category of $\mathbb{Z}$-modules, the morphism

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

is a quasi-isomorphism; but it clearly has no inverse, not even up to homotopy, because there are no nontrivial homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$.

Let me now explain the classical construction of the derived category. Let $\mathfrak{A}$ be an arbitrary abelian category (such as modules over a ring, or coherent sheaves on a scheme). Depending on what kind of complexes we want to consider, there are several derived categories: the unbounded derived category $\mathrm{D}(\mathfrak{A})$, whose objects are arbitrary complexes of objects in $\mathfrak{A}$; the categories $\mathrm{D}^+(\mathfrak{A})$ and $\mathrm{D}^-(\mathfrak{A})$, whose objects are semi-infinite complexes that are allowed to be infinite in the positive respectively negative direction; and finally the bounded derived category $\mathrm{D}^b(\mathfrak{A})$, whose objects are bounded complexes of objects in $\mathfrak{A}$. All of these categories are constructed in two stages; we explain this in the case of $\mathrm{D}^b(\mathfrak{A})$.

(1) Starting from the category of bounded complexes $\mathrm{K}^b(\mathfrak{A})$, form the so-called homotopy category $\mathrm{H}^b(\mathfrak{A})$. It has exactly the same objects, but the morphisms between two complexes are taken up to homotopy; in other words,

$$\mathrm{Hom}_{\mathrm{H}^b(\mathfrak{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{K}^b(\mathfrak{A})}(A^\bullet, B^\bullet) \big/ \mathrm{Hom}^0_{\mathrm{K}^b(\mathfrak{A})}(A^\bullet, B^\bullet),$$

where $\mathrm{Hom}^0_{\mathrm{K}^b(\mathfrak{A})}(A^\bullet, B^\bullet)$ denotes the subgroup of those morphisms that are homotopic to zero.

(2) Now form the derived category $\mathrm{D}^b(\mathfrak{A})$ by inverting quasi-isomorphisms; this can be done by a formal construction similar to the passage from $\mathbb{Z}$ to $\mathbb{Q}$. That is to say, in $\mathrm{D}^b(\mathfrak{A})$, a morphism between two complexes $A^\bullet$ and $B^\bullet$ is represented by a fraction f/h, which stands for the diagram

$$\begin{tikzcd} & C^\bullet \arrow[dl, "h"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $f\colon C^\bullet \to B^\bullet$ is a morphism of complexes and $h\colon C^\bullet \to A^\bullet$ is a quasi-isomorphism. As with ordinary fractions, there is an equivalence relation that we shall not dwell on; it is also not entirely trivial to show that the composition of two morphisms is again a morphism.

In other words, the objects of the derived category are still just complexes; but the set of morphisms between two complexes has become more complicated (especially because a morphism may involve an additional complex).

Example 16.5. For us, the most interesting case is when the abelian category is $\mathrm{Coh}(X)$, the category of coherent sheaves on a scheme X. By applying the above construction, we get the bounded derived category $\mathrm{D}^b\big(\mathrm{Coh}(X)\big)$; once again, the objects of this category are just bounded complexes of coherent sheaves. For practical purposes, a broader definition of the derived category is more useful. Inside the unbounded derived category $\mathrm{D}(\mathscr{O}_X)$ of all complexes of sheaves of $\mathscr{O}_X$-modules, consider the full subcategory $\mathrm{D}^b_{coh}(\mathscr{O}_X)$; by definition, a complex

$$\cdots \to \mathscr{F}^{-1} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to \cdots$$

belongs to this subcategory if its cohomology sheaves $\mathcal{H}^i(\mathscr{F}^\bullet)$ are coherent, and nonzero for only finitely many values of i. Clearly,

$$\mathrm{D}^b\big(\mathrm{Coh}(X)\big) \subseteq \mathrm{D}^b_{coh}(\mathscr{O}_X),$$

and under some mild assumptions on X, this inclusion is actually an equivalence of categories. The larger category has the advantage of being more flexible: for example, an injective resolution of a coherent sheaf is an object of $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ but not of $\mathrm{D}^b\big(\mathrm{Coh}(X)\big)$.

Morphisms in the derived category. The definition of the derived category leads to several questions. The first one is whether one can describe the space of morphisms between two complexes in more basic terms. At least in the case of complexes with only one nonzero cohomology object, this is possible.

We first define the following *shift functor*. Given a complex $A^\bullet \in \mathrm{K}(\mathfrak{A})$ and an integer $n \in \mathbb{Z}$, we obtain a new complex $A^\bullet[n]$ by setting

$$A^\bullet[n] = A^{\bullet+n};$$

we also multiply all the differentials in the original complex by the factor $(-1)^n$. (This convention makes it easier to remember certain formulas.) For example, if $A^\bullet$ is the complex

$$\cdots \longrightarrow A^{-1} \xrightarrow{d} A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \longrightarrow \cdots$$

then $A^\bullet[1]$ is the same complex shifted to the left by one step,

$$\cdots \longrightarrow A^0 \xrightarrow{-d} A^1 \xrightarrow{-d} A^2 \xrightarrow{-d} A^3 \longrightarrow \cdots$$

and with the sign of all differentials changed. This operation passes to the derived category, and defines a collection of functors $[n]\colon D(\mathfrak{A}) \to D(\mathfrak{A})$.

Example 16.6. Morphisms in $\mathrm{D}^b(\mathfrak{A})$ are related to Ext-groups in the sense of Yoneda. (When the abelian category $\mathfrak{A}$ has enough injective objects, these are the same as the derived functors of Hom, computed using an injective resolution.) If A and B are two objects of the abelian category $\mathfrak{A}$, then one has

$$\mathrm{Hom}_{\mathrm{D}^b(\mathfrak{A})}\big(A, B[n]\big) \simeq \mathrm{Ext}^n(A, B);$$

in particular, this group is trivial for $n < 0$. Let us consider the case $n = 1$. An element of $\mathrm{Ext}^1(A, B)$ is represented by a short exact sequence of the form

$$0 \to B \to E \to A \to 0.$$

Now the morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

is obviously a quasi-isomorphism; on the other hand, we have

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

and together, they determine a morphism in $\mathrm{D}^b(\mathfrak{A})$ from A (viewed as a complex in degree 0) to $B[1]$ (viewed as a complex in degree -1).

Exercise 16.1. Show that, conversely, every element of $\operatorname{Hom}_{\mathrm{D}^b(\mathfrak{A})}\big(A, B[1]\big)$ gives rise to an extension of A by B, and that the two constructions are inverse to each other.

Other models for the derived category. Recall that the objects of the bounded derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ are complexes of sheaves of $\mathscr{O}_X$-modules whose cohomology sheaves are coherent and vanish outside some bounded interval. I already mentioned that, under some mild assumptions on X, this category is equivalent to the much smaller category $\mathrm{D}^b\big(\mathrm{Coh}(X)\big)$, whose objects are bounded complexes of coherent sheaves on X. There are various other models for the derived category, each based on a certain class of sheaves (such as injective sheaves or flat sheaves). Let me illustrate this principle with the following example.

Example 16.7. Let $\mathrm{Inj}(\mathscr{O}_X)$ denote the (additive, but not abelian) category of injective sheaves of $\mathscr{O}_X$-modules. Every $\mathscr{O}_X$-module has a semi-infinite resolution by injectives; using the Cartan-Eilenberg construction, every semi-infinite complex of $\mathscr{O}_X$-modules is quasi-isomorphic to a semi-infinite complex of injectives. This means that the inclusion

$$\mathrm{D}^+\big(\mathrm{Inj}(\mathscr{O}_X)\big) \subseteq \mathrm{D}^+(\mathscr{O}_X)$$

is an equivalence of categories. By restricting to complexes with bounded and coherent cohomology sheaves, we also obtain an equivalence of categories

$$\mathrm{D}^b_{coh}\big(\mathrm{Inj}(\mathscr{O}_X)\big) \simeq \mathrm{D}^b_{coh}(\mathscr{O}_X).$$

The advantage of using injectives is that we do not need to worry about inverses for quasi-isomorphisms. Indeed, suppose that $f\colon I_1^\bullet \to I_2^\bullet$ is a quasi-isomorphism between two complexes of injective $\mathscr{O}_X$-modules. The universal mapping property of injectives implies that there is a morphism of complexes $g\colon I_2^\bullet \to I_1^\bullet$ such that both $f \circ g$ and $g \circ f$ are homotopic to the identity. Thus

$$\mathrm{D}^+\big(\mathrm{Inj}(\mathscr{O}_X)\big) \simeq H^+\big(\mathrm{Inj}(\mathscr{O}_X)\big)$$

and, extending our earlier notation in the obvious way, also

$$H^b_{coh}\big(\mathrm{Inj}(\mathscr{O}_X)\big) \simeq \mathrm{D}^b_{coh}(\mathscr{O}_X).$$

The same construction works for sheaves of flat $\mathscr{O}_X$-modules; under certain additional assumptions on the scheme X, one can also use locally free sheaves.

In this model for the derived category, the morphisms are much easier to describe. Nevertheless, it is better to work with the category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$, because it gives us more flexibility: we can choose injective or flat or locally free resolutions as the occasion demands.

Triangulated categories. The derived category is no longer an abelian category, because the kernel and cokernel of a morphism do not make sense. (This is due to all the additional morphisms that we have introduced when adding inverses for quasi-isomorphisms.) But there is a replacement for short exact sequences, the so-called distinguished triangles, and $D^b(\mathfrak{A})$ is an example of a *triangulated category*.

A triangulated category is given by specifying a class of triangles. The motivation for introducing triangles lies in the mapping cone construction from homological algebra; let us briefly review this construction, and explain in what sense it acts as a substitute for short exact sequences. Given a morphism of complexes $f\colon A^\bullet \to B^\bullet$, the *mapping cone* of f is the complex

$$C_f^\bullet = B^\bullet \oplus A^\bullet[1] = B^\bullet \oplus A^{\bullet+1}$$

with differential $d(b,a) = (db + fa, -da)$. (The terminology comes from the mapping cone in algebraic topology.) Since we defined $A^\bullet[1]$ by changing the sign of all differentials, this makes the sequence of complexes

$$0 \to B^\bullet \to C_f^\bullet \to A^\bullet[1] \to 0$$

short exact. In total, we have a sequence of four morphisms

$$A^\bullet \to B^\bullet \to C_f^\bullet \to A^\bullet[1], \tag{16.8}$$

and the composition of any two adjacent morphisms is zero *up to homotopy.*

Exercise 16.2. Verify that the composite morphisms

$$A^\bullet \to B^\bullet \to C_f^\bullet \qquad \text{and} \qquad C_f^\bullet \to A^\bullet[1] \to B^\bullet[1]$$

are both homotopic to zero.

A sequence of four morphisms as in (16.8) is called a *triangle*; this is because we can arrange it into the shape of a triangle, with the convention that the arrow marked [1] really goes from $C_f^\bullet$ to $A^\bullet[1]$:

$$\begin{tikzcd} & A^\bullet \arrow[dr] & \\ C_f^\bullet \arrow[ur, "{[1]}"] & & B^\bullet \arrow[ll] \end{tikzcd}$$

The short exact sequence of complexes gives rise to a long exact sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C_f^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

for the cohomology of the complexes. In order to write down this long exact sequence, all we need is the four morphisms in (16.8). Taking this example as a model, we say that any sequence of four morphisms of complexes

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

is a *distiguished triangle* if it is isomorphic, in the derived category, to a triangle coming from the mapping cone construction. (In particular, the composition of two adjacent morphisms in the triangle is then actually homotopic to zero.) This definition endows the derived category with the structure of a triangulated category.

Here are two basic properties of distinguished triangles that you should try to verify as an exercise. There are many others, and by abstracting from this example, Verdier arrived at the concept of a triangulated category; since the precise definition is not relevant for our purposes, we shall not dwell on the details.

Exercise 16.3. Suppose that $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle. Show that $B^\bullet \to C^\bullet \to A^\bullet[1] \to B^\bullet[1]$ and $C^\bullet[-1] \to A^\bullet \to B^\bullet \to C^\bullet$ are again distinguished triangles. This means that distinguished triangles can be "rotated" in both directions.

Exercise 16.4. Show that a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ gives rise to a long exact sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

in the abelian category $\mathfrak{A}$.

I already mentioned that distinguished triangles are a replacement for short exact sequences; let me elaborate on this point a bit. On the one hand, the prototypical example of a distinguished triangle in (16.8) came from the short exact sequence of the mapping cone. On the other hand, once we look at complexes up to quasi-isomorphism, every short exact sequence of complexes is actually that of a mapping cone (under some conditions on $\mathfrak{A}$). Let me illustrate this claim with the example of modules over a ring.

Example 16.9. Suppose we have a short exact sequence of complexes of R-modules

$$0 \to B^\bullet \to C^\bullet \to A^\bullet[1] \to 0.$$

Up to quasi-isomorphism, we can replace any complex by a free resolution, and so we may assume that $A^\bullet$ is a complex of free R-modules. We can then choose splittings

$$C^n \simeq B^n \oplus A^{n+1}.$$

With respect to this decomposition, the differential $d\colon C^n \to C^{n+1}$ is represented by a matrix

$$\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}$$

for some homomorphism $f\colon A^n \to B^n$; the identity $d \circ d = 0$ implies that f defines a morphism of complexes from $A^\bullet$ to $B^\bullet$, and our exact sequence of complexes is the one for the mapping cone of f.

In closing, let me mention one other general fact that is frequently useful. Namely, suppose that $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle in $\mathrm{D}^b(\mathfrak{A})$. Then for every $E^\bullet \in \mathrm{D}^b(\mathfrak{A})$, one gets two long exact sequences of abelian groups

$$\cdots \to \operatorname{Hom}(E^\bullet, A^\bullet) \to \operatorname{Hom}(E^\bullet, B^\bullet) \to \operatorname{Hom}(E^\bullet, C^\bullet) \to \operatorname{Hom}\bigl(E^\bullet, A^\bullet[1]\bigr) \to \cdots$$
$$\cdots \to \operatorname{Hom}\bigl(A^\bullet[1], E^\bullet\bigr) \to \operatorname{Hom}(C^\bullet, E^\bullet) \to \operatorname{Hom}(B^\bullet, E^\bullet) \to \operatorname{Hom}(A^\bullet, E^\bullet) \to \cdots$$

where $\operatorname{Hom}(-,-)$ means the set of morphisms in $\mathrm{D}^b(\mathfrak{A})$.

Derived functors. From now on, we shall concentrate on the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$, where X is a scheme. Here is a very useful fact:

Example 16.10. If X is nonsingular and quasi-compact, so that every coherent sheaf on X has a finite resolution by locally free sheaves, then every complex in $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ is quasi-isomorphic to a bounded complex of locally free sheaves.

Our goal is to define derived functors for the commonly used functors in algebraic geometry, such as $\otimes$, $\mathcal{H}om$, or pushforwards and pullbacks. The original functors are either left or right exact, and in classical homological algebra, the higher derived functors correct the lack of exactness. In the setting of triangulated categories, the relevant definition is the following.

Definition 16.11. An additive functor between two triangulated categories is *exact* if it takes distinguished triangles to distinguished triangles.

If we have an exact functor $F\colon \mathrm{D}^b(\mathfrak{A}) \to \mathrm{D}^b(\mathfrak{B})$ between the derived categories of two abelian categories, we get a long exact sequence in cohomology: if

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

is a distinguished triangle in $\mathrm{D}^b(\mathfrak{A})$, then

$$F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet) \to F(A^\bullet)[1]$$

is a distinguished triangle in $\mathrm{D}^b(\mathfrak{B})$, and so

$$\cdots \to H^i F(A^\bullet) \to H^i F(B^\bullet) \to H^i F(C^\bullet) \to H^{i+1} F(A^\bullet) \to \cdots$$

is a long exact sequence in the abelian category $\mathfrak{B}$. This explains the terminology.

When defining a derived functor, we have two choices:

(1) Use a definition that works only for certain complexes, such as complexes of injective sheaves or flat sheaves. Then show that the subcategory consisting of such complexes is equivalent to the entire derived category. In this way, we obtain a non-constructive definition of the functor.

(2) Use a definition that works for arbitrary complexes. This may require more effort, but seems better from a mathematical point of view.

Example 16.12. Let $f\colon X \to Y$ be a morphism of schemes, say quasi-compact and separated (in order for f_* to preserve quasi-coherence). We want to define the derived functor $\mathbf{R}f_*\colon \mathrm{D}^+\big(\mathrm{QCoh}(X)\big) \to \mathrm{D}^+\big(\mathrm{QCoh}(Y)\big)$. Since we already know that injective sheaves are acyclic, we should obviously define

$$\mathbf{R}f_* I^\bullet = f_* I^\bullet$$

if $I^\bullet$ is a complex of injective sheaves. Since the subcategory $\mathrm{D}^+\big(\mathrm{Inj}(X)\big)$ is equivalent to $\mathrm{D}^+\big(\mathrm{QCoh}(X)\big)$, we can choose an inverse functor to the inclusion – this basically amounts to choosing an injective resolution for every complex of quasi-coherent sheaves – and compose the two. In this way, we obtain a functor

$$\mathbf{R}f_*\colon \mathrm{D}^+\big(\mathrm{QCoh}(X)\big) \to \mathrm{D}^+\big(\mathrm{QCoh}(Y)\big).$$

If f is proper, then f_* preserves coherence, and $\mathbf{R}f_*$ restricts to a functor

$$\mathbf{R}f_*\colon \mathrm{D}^b_{coh}(\mathscr{O}_X) \to \mathrm{D}^b_{coh}(\mathscr{O}_Y).$$

It remains to verify that $\mathbf{R}f_*$ is an exact functor.

Exercise 16.5. Show that $\mathbf{R}f_*$ takes distinguished triangles to distinguished triangles. (Hint: It is enough to prove this for a triangle of the form

$$I_1^\bullet \to I_2^\bullet \to C_\varphi^\bullet \to I_1^\bullet[1],$$

for $\varphi\colon I_1^\bullet \to I_2^\bullet$ a morphism between two complexes of injective sheaves.)

Example 16.13. If the above definition of $\mathbf{R}f_*$ involves too many choices for your taste, here is another possibility. Flasque sheaves are also acyclic for f_*, and have the advantage that there is a canonical resolution by flasque sheaves, the so-called *Godement resolution*. Given a sheaf of abelian groups $\mathscr{F}$, let $G^0(\mathscr{F})$ denote the sheaf of discontinuous sections: for any open subscheme $U \subseteq X$,

$$G^0(\mathscr{F})(U) = \prod_{x \in U} \mathscr{F}_x.$$

This sheaf is flasque and contains $\mathscr{F}$ as a subsheaf. Now we define $G^1(\mathscr{F})$ by applying the same construction to the cokernel of $\mathscr{F} \hookrightarrow G^0(\mathscr{F})$; in general, we set $G^{n+1}(\mathscr{F}) = G^0\big(G^n(\mathscr{F})/G^{n-1}(\mathscr{F})\big)$. The resulting complex of sheaves

$$0 \to \mathscr{F} \to G^0(\mathscr{F}) \to G^1(\mathscr{F}) \to G^2(\mathscr{F}) \to \cdots$$

is exact; this is the Godement resolution $G^\bullet(\mathscr{F})$. The same construction produces canonical flasque resolutions for complexes of sheaves: apply the construction to each sheaf in the complex to get a double complex, and then take the associated single complex. This allows us to define $\mathbf{R}f_*$ by setting

$$\mathbf{R}f_* F = f_* G^\bullet(F)$$

for any $F \in \mathrm{D}^+(\mathscr{O}_X)$. One can show that $\mathbf{R}f_*\mathscr{F}$ is canonically isomorphic to $f_*\mathscr{F}$ when $\mathscr{F}$ is a flasque sheaf; up to isomorphism, the two constructions of $\mathbf{R}f_*$ are therefore the same.

By one of those methods, one can also define the derived functors $\overset{\mathbf{L}}{\otimes}$, $\mathbf{R}\mathcal{H}om$, $\mathbf{R}\Gamma$, $\mathbf{R}\mathrm{Hom}$, as well as $\mathbf{L}f^*$ for morphisms $f\colon X \to Y$. All of the properties of the underived functors carry over to this setting: for example, $\mathbf{L}f^*$ is the left adjoint of $\mathbf{R}f_*$. In classical homological algebra, the composition of two functors leads to a spectral sequence (such as the Grothendieck spectral sequence); in the derived category, this simply becomes an identity between two derived functors.

Example 16.14. For two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, one has $\mathbf{R}g_* \circ \mathbf{R}f_* \simeq \mathbf{R}(g \circ f)_*$. This can be proved by observing that the pushforward of an injective sheaf is again injective: for a complex of injective sheaves,

$$(g \circ f)_* I^\bullet = g_*\big(f_* I^\bullet\big).$$

A special case of this is the formula $\mathbf{R}\Gamma(Y,-) \circ \mathbf{R}f_* \simeq \mathbf{R}\Gamma(X,-)$, which is the derived category version of the Leray spectral sequence.

Example 16.15. Similar reasoning proves the formula $\mathbf{R}\Gamma \circ \mathbf{R}\mathcal{H}om \simeq \mathbf{R}\mathrm{Hom}$.

The big advantage of working in the derived category is that many relations among the underived functors that are true only for locally free sheaves, now hold in general. Technically, this is true on nonsingular varieties, because every complex in $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ is then quasi-isomorphic to a bounded complex of locally free sheaves.

As a case in point, let us consider the projection formula. The version in Hartshorne says that if $f\colon X \to Y$ is a morphism of schemes, and if $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module of finite rank, then $f_*\big(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E}\big) \simeq f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E}$. In the derived category, we have the following generalization.

Proposition 16.16. *Let $f\colon X \to Y$ be a morphism of schemes, with Y nonsingular and quasi-compact. Then one has*

$$\mathbf{R}f_*\big(F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \mathbf{L}f^*G\big) \simeq \mathbf{R}f_*F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Y} G$$

for every $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and every $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

Proof. We may assume without loss of generality that G is a bounded complex of locally free sheaves and that F is a complex of injective sheaves. In that case,

$$\mathbf{R}f_*\big(F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \mathbf{L}f^*G\big) = f_*\big(F \otimes_{\mathscr{O}_X} f^*G\big),$$

and by the usual projection formula, this is isomorphic to

$$f_*F \otimes_{\mathscr{O}_Y} G = \mathbf{R}f_*F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Y} G. \qquad \square$$