Lecture 16: April 8

Non-characteristic pullback and coherence. Recall that if $f\colon X \to Y$ is a morphism between nonsingular algebraic varieties, we have the following morphisms between cotangent bundles:

$$\begin{array}{ccc} X \times_Y T^*Y & \xrightarrow{df} & T^*X \\ \downarrow{\scriptstyle p_2} & & \\ T^*Y & & \end{array} \tag{16.1}$$

We said last time that a coherent left $\mathscr{D}_Y$-module $\mathcal{M}$ is called *non-characteristic* with respect to f if $p_2^{-1}\operatorname{Ch}(\mathcal{M})$ is finite over its image in T^*X (under the morphism df). Here are three typical examples.

Example 16.2. If f is a smooth morphism, then df is a closed embedding, and so every coherent left $\mathscr{D}_Y$-module is noncharacteristic with respect to f.

Example 16.3. If $\mathcal{M}$ is a vector bundle with integrable connection, then $\operatorname{Ch}(\mathcal{M})$ is the zero section in T^*Y. Since the zero section in $X \times_Y T^*Y$ and in T^*X are both isomorphic to X, the restriction of df to $p_2^{-1}\operatorname{Ch}(\mathcal{M})$ is an isomorphism, and so $\mathcal{M}$ is non-characteristic with respect to any morphism f. So being non-characteristic is really a condition on the other components of the characteristic variety.

Example 16.4. The left $\mathscr{D}_Y$-module $\mathscr{D}_Y$ is *never* non-characteristic with respect to a closed embedding $f\colon X \hookrightarrow Y$ (as long as $\dim X < \dim Y$). Indeed, $\operatorname{Ch}(\mathcal{M}) = T^*Y$ in this case, and since df has positive-dimensional fibers, $p_2^{-1}\operatorname{Ch}(\mathcal{M})$ is not finite over its image.

Our goal for today is to show that pulling back preserves coherence in the non-characteristic setting.

Theorem 16.5. *Let $f\colon X \to Y$ be a morphism between nonsingular algebraic varieties, and $\mathcal{M}$ a coherent left $\mathscr{D}_Y$-module. If $\mathcal{M}$ is non-characteristic with respect to f, then the following is true.*
- *(a) The pullback $f^*\mathcal{M}$ is a coherent left $\mathscr{D}_X$-module.*
- *(b) One has $L^{-j}f^*\mathcal{M} = 0$ for $j \geq 1$.*
- *(c) One has $\operatorname{Ch}(f^*\mathcal{M}) = df\bigl(p_2^{-1}\operatorname{Ch}(\mathcal{M})\bigr)$.*

Note that since $df\colon p_2^{-1}\operatorname{Ch}(\mathcal{M}) \to T^*X$ is a finite morphism, the image is again a closed algebraic subset of T^*X. Thus the statement in (c) makes sense.

For the proof, the idea is to factor $f\colon X \to Y$ as a closed embedding followed by a smooth morphism, and to analyze the two cases separately.

Smooth morphisms. Suppose that $f\colon X \to Y$ is a smooth morphism. In the diagram in (16.1), the morphism p_2 is then also smooth, and the morphism df is a closed embedding. Now let $\mathcal{M}$ be a coherent left $\mathscr{D}_Y$-module. We have

$$f^*\mathcal{M} = \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}\mathcal{M} \cong \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathcal{M},$$

and since smooth morphisms are flat, the tensor product with $\mathscr{O}_X$ is exact. In particular, the higher derived functors of the tensor product are zero, and so $L^{-j}f^*\mathcal{M} = 0$ for $j \geq 1$. This proves (b). Next, we show that $f^*\mathcal{M}$ is coherent over $\mathscr{D}_X$. By assumption, $\mathcal{M}$ is coherent over $\mathscr{D}_Y$, and so $f^{-1}\mathcal{M}$ is coherent over $f^{-1}\mathscr{D}_Y$. Since the left $\mathscr{D}_X$-module structure on $f^*\mathcal{M}$ comes from $\mathscr{D}_{X\to Y}$, it is therefore enough to show that the morphism

$$\mathscr{D}_X \to \mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y, \quad P \mapsto P \cdot (1 \otimes 1)$$

is surjective. This can be done locally. We can therefore assume that X and Y are affine, and we can choose local coordinates $x_1, \dots, x_{n+r} \in \Gamma(X, \mathscr{O}_X)$ and $y_1, \dots, y_n \in \Gamma(Y, \mathscr{O}_Y)$, in such a way that the morphism on tangent sheaves

$$\mathscr{T}_X \to f^* \mathscr{T}_Y = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{T}_Y$$

maps ∂_{x_j} to $1 \otimes \partial_{y_j}$ for $1 \leq j \leq n$, and to zero otherwise. (This means that $\partial_{x_{n+1}}, \dots, \partial_{x_{n+r}}$ generate the relative tangent sheaf $\mathscr{T}_{X/Y}$.) Now every element of $\Gamma(X, \mathscr{D}_{X \to Y})$ can be written in the form

$$\sum_{\alpha \in \mathbb{N}^n} g_\alpha \otimes \partial_{y_1}^{\alpha_1} \cdots \partial_{y_n}^{\alpha_n},$$

with $g_\alpha \in \Gamma(X, \mathscr{O}_X)$, and because of how we defined the $\mathscr{D}_X$-module structure on the transfer module, this expression equals

$$\sum_{\alpha \in \mathbb{N}^n} g_\alpha \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \cdot (1 \otimes 1).$$

Thus $\mathscr{D}_X \to \mathscr{D}_{X \to Y}$ is indeed surjective, with kernel generated by the relative tangent sheaf $\mathscr{T}_{X/Y}$.

It remains to prove that $\operatorname{Ch}(f^*\mathcal{M}) = df\big(p_2^{-1} \operatorname{Ch}(\mathcal{M})\big)$. Choose a good filtration $F_\bullet \mathcal{M}$, and observe that because f is flat, we have $f^* F_j \mathcal{M} \subseteq f^* \mathcal{M}$. If we set $\mathscr{N} = f^*\mathcal{M}$, we thus get a filtration with terms $F_j \mathscr{N} = f^* F_j \mathcal{M}$. It is clear that each $F_j \mathscr{N}$ is a coherent $\mathscr{O}_X$-module; moreover, flatness of f gives

$$\operatorname{gr}_j^F \mathscr{N} = F_j \mathscr{N} / F_{j-1} \mathscr{N} \cong f^* \operatorname{gr}_j^F \mathcal{M}.$$

Once we check that $F_\bullet \mathscr{N}$ is a good filtration, we can use it to compute $\operatorname{Ch}(\mathscr{N})$. Working locally, we can assume that X and Y are affine, and that we have local coordinates $x_1, \dots, x_{n+r} \in \Gamma(X, \mathscr{O}_X)$ and $y_1, \dots, y_n \in \Gamma(Y, \mathscr{O}_Y)$ as above. To abbreviate, set $A = \Gamma(X, \mathscr{O}_X)$ and $B = \Gamma(Y, \mathscr{O}_Y)$; then A is a smooth B-algebra. We shall use the same symbol ∂_j to denote both ∂_{x_j} and ∂_{y_j}; then the morphism on tangent sheaves takes ∂_j to $1 \otimes \partial_j$ for $1 \leq j \leq n$, and to zero otherwise.

Let us set $M = \Gamma(Y, \mathcal{M})$ and $N = \Gamma(X, \mathscr{N})$. By construction,

$$N = A \otimes_B M \quad \text{and} \quad F_j N = A \otimes_B F_j M \quad \text{and} \quad \operatorname{gr}_j^F N = A \otimes_B \operatorname{gr}_j^F M.$$

As the filtration on M is good, the associated graded $\operatorname{gr}^F M$ is finitely generated over $\operatorname{gr}^F D(B) = B[\partial_1, \dots, \partial_n]$. The left $D(A)$-module structure on N is given by

$$\partial_j (a \otimes m) = \begin{cases} \partial_j a \otimes m + a \otimes \partial_j m & \text{if } 1 \leq j \leq n, \\ \partial_j a \otimes m & \text{if } n+1 \leq j \leq n+r. \end{cases}$$

This formula shows that the filtration $F_\bullet N$ is compatible with the action by $D(A)$. It also shows that $\partial_{n+1}, \dots, \partial_{n+r}$ act trivially on

$$\operatorname{gr}^F N = A \otimes_B \operatorname{gr}^F M,$$

and that $\partial_1, \dots, \partial_n$ only act on the second factor. Said differentialy, we have an isomorphism of graded $A[\partial_1, \dots, \partial_{n+r}]$-modules

$$\operatorname{gr}^F N \cong A[\partial_1, \dots, \partial_n] \otimes_{B[\partial_1, \dots, \partial_n]} \operatorname{gr}^F M, \tag{16.6}$$

with $A[\partial_1, \dots, \partial_{n+r}]$ acting on the first factor in the obvious way. This says that $\operatorname{gr}^F N$ is finitely generated over $A[\partial_1, \dots, \partial_{n+r}]$, and so $F_\bullet N$ is a good filtration.

It is now easy to compute the characteristic variety $\mathrm{Ch}(\mathcal{N})$. If we rewrite the diagram in (16.1) in terms of rings, we get

$$\begin{tikzcd}
\operatorname{Spec} A[\partial_1, \dots, \partial_n] \arrow[r, "df"] \arrow[d, "p_2"] & \operatorname{Spec} A[\partial_1, \dots, \partial_{n+r}] \\
\operatorname{Spec} B[\partial_1, \dots, \partial_n]
\end{tikzcd}$$

with p_2 induced by the morphism of rings $B \to A$, and df induced by the quotient morphism $A[\partial_1, \dots, \partial_{n+r}] \to A[\partial_1, \dots, \partial_n]$. Thus (16.6) says that the coherent sheaf on $T^*X = \operatorname{Spec} A[\partial_1, \dots, \partial_{n+r}]$ corresponding to $\mathrm{gr}^F N$ is obtained by first pulling back $\mathrm{gr}^F M$ along p_2, and then pushing forward along df. Globally,

$$\widetilde{\mathrm{gr}^F \mathcal{N}} \cong df_* p_2^* \widetilde{\mathrm{gr}^F \mathcal{M}},$$

and since p_2 is surjective and df a closed embedding, we get

$$\mathrm{Ch}(\mathcal{N}) = df\big(p_2^{-1}\,\mathrm{Ch}(\mathcal{M})\big),$$

proving (c) for all smooth morphisms.

Factorizing through the graph. Using the graph embedding, we can write any morphism $f\colon X \to Y$ as the composition of a closed embedding $i\colon X \hookrightarrow Z$ and a smooth morphism $g\colon Z \to Y$. (Here $Z = X \times Y$, of course, but let me write Z to simplify the notation.) We already know that $\mathcal{N} = g^*\mathcal{M}$ is coherent over $\mathscr{D}_Z$, and that $\mathrm{Ch}(\mathcal{N}) = dg\big(p_2^{-1}\,\mathrm{Ch}(\mathcal{M})\big)$. Using the big diagram

$$\begin{tikzcd}
X \times_Y T^*Y \arrow[r, hook] \arrow[d, "i \times \mathrm{id}"] \arrow[rr, bend left, "df"] & X \times_Z T^*Z \arrow[r, "di"] \arrow[d, "p_2"] & T^*Z \\
Z \times_Y T^*Y \arrow[r, hook, "dg"] \arrow[d, "p_2"] & T^*Z & \\
T^*Y & &
\end{tikzcd}$$

from last time, we see that $p_2^{-1}\,\mathrm{Ch}(\mathcal{N})$ is finite over its image in T^*X (under the morphism di); this says that $\mathcal{N}$ is non-characteristic with respect to the closed embedding $i\colon X \hookrightarrow Z$. As $f^*\mathcal{M} \cong i^*\mathcal{N}$, this reduces the proof of Theorem 16.5 to the case of a closed embedding.

Closed embeddings. Suppose now that $f\colon X \to Y$ is a closed embedding. We are only going to treat the case where $\dim X = \dim Y - 1$; to go from there to the general case, one uses the fact that f can be locally factored as a composition of $\dim Y - \dim X$ closed embeddings of codimension one (because closed embeddings between nonsingular algebraic varieties are locally complete intersections).

The problem is local, and so we can assume that Y is affine, with $B = \Gamma(Y, \mathscr{O}_Y)$. Choose local coordinates $y_0, y_1, \dots, y_n \in B$, in such a way that X is defined by the equation $y_0 = 0$; then $A = \Gamma(X, \mathscr{O}_X) \cong B/By_0$, and the images $x_1, \dots, x_n \in A$ of $y_1, \dots, y_n \in B$ are local coordinates on X. The morphism on tangent sheaves

$$\mathscr{T}_X \to f^*\mathscr{T}_Y = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{T}_Y$$

takes ∂_{x_j} to $1 \otimes \partial_{y_j}$ for $1 \le j \le n$. (The remaining vector field ∂_{y_0} is not in the image; it generates the normal bundle.) We again write ∂_j for both ∂_{x_j} and ∂_{y_j}, so that the morphism on tangent sheaves takes ∂_j to $1 \otimes \partial_j$. With this notation, the diagram in (16.1) becomes

$$\begin{tikzcd} \operatorname{Spec} A[\partial_0, \dots, \partial_n] \arrow[r, "df"] \arrow[d, "p_2"] & \operatorname{Spec} A[\partial_1, \dots, \partial_n] \\ \operatorname{Spec} B[\partial_0, \dots, \partial_n]. & \end{tikzcd} \tag{16.7}$$

This time, p_2 is a closed embedding and df is smooth of relative dimension one.

We are going to use the following basic fact from algebraic geometry.

Lemma 16.8. *Let B be a finitely generated A-algebra.*

(1) If B is integral over A, then every finitely generated B-module M is also finitely generated as an A-module.
(2) If M is a finitely generated B-module such that $\operatorname{Supp} M$ is finite over $\operatorname{Spec} A$, then M is also finitely generated as an A-module.

Proof. The first assertion follows from the fact that B itself is finitely generated as an A-module. To prove the second assertion, we may replace B by the quotient ring $B/\operatorname{Ann}_B(M)$ and assume without loss of generality that $\operatorname{Ann}_B(M) = 0$. The support of M is then the reduced closed subscheme defined by the nilradical of B, and so the hypothesis says that $B/\operatorname{Nil} B$ is integral over A. This means that for every $b \in B$, there is a monic polynomial $h(t) \in A[t]$ such that $h(b) \in \operatorname{Nil} B$. But then $h(b)^m = 0$ for some $m \geq 1$, and so b is integral over A. We now conclude from the first assertion that M is finitely generated as an A-module. □

Now let $\mathcal{M}$ be a coherent left $\mathscr{D}_Y$-module that is non-characteristic with respect to f. Set $M = \Gamma(Y, \mathcal{M})$, which is a finitely generated module over the ring of differential operators $D(B) = \Gamma(Y, \mathscr{D}_Y)$. The following lemma expresses the non-characteristic property of M in terms of differential operators.

Lemma 16.9. *For every $u \in M$, there exists a nontrivial differential operator $P \in D(B)$ that is non-characteristic with respect to $y_0 = 0$ and satisfies $Pu = 0$.*

Proof. The submodule $D(B)u \subseteq M$ is isomorphic to $D(B)/I$, where

$$I = \left\{ P \in D(B) \;\middle|\; Pu = 0 \right\}$$

is a left ideal in $D(B)$. The characteristic variety of $D(B)/I$ is contained in that of M, and so $D(B)/I$ is again non-characteristic with respect to f. As a subset of $T^*Y = \operatorname{Spec} B[\partial_0, \dots, \partial_n]$, the characteristic variety of $D(B)/I$ is cut out by the principal symbols $\sigma(P) \in B[\partial_0, \dots, \partial_n]$ of all the differential operators $P \in I$. Its preimage under p_2 is therefore cut out by their images in $A[\partial_0, \dots, \partial_n]$. Because this subset is finite over $\operatorname{Spec} A[\partial_1, \dots, \partial_n]$, we can argue as in the preceding lemma to show that there is a monic polynomial $h(t)$ of some degree $d \geq 1$, with coefficients in the ring $A[\partial_1, \dots, \partial_n]$, such that $h(\partial_0) \in A[\partial_0, \dots, \partial_n]$ belongs to the ideal generated by $\sigma(P)$ for $P \in I$. Keeping all terms in $h(\partial_0)$ that are homogeneous of degree d, we conclude that there exists a differential operator $P \in I$ of order d, such that the image of $\sigma(P)$ in $A[\partial_0, \dots, \partial_n]$ contains the term ∂_0^d. But this says exactly that P is non-characteristic with respect to $y_0 = 0$. □

Note. Since M is finitely generated over $D(B)$, the lemma implies that there exist finitely many differential operators $P_1, \dots, P_r \in D(B)$, all non-characteristic with respect to $y_0 = 0$, and a surjective morphism

$$\bigoplus_{i=1}^{r} D(B)/D(B)P_i \to M.$$

By applying the same observation to the kernel, one can in fact show that M admits a resolution by non-characteristic $D(B)$-modules of the form $D(B)/D(B)P$.

Now let us continue with the proof of Theorem 16.5. The derived functors $L^{-j} f^* \mathcal{M}$ are computed, in our local coordinates, by the complex of $D(A)$-modules

$$M \xrightarrow{y_0} M.$$

To show that $L^{-j} f^* \mathcal{M} = 0$ for every $j \geq 1$, we only have to argue that multiplication by y_0 is injective. Suppose that we have some $u \in M$ with $y_0 u = 0$. By the lemma, we can find a differential operator $P \in D(B)$, say of degree $d \geq 0$, such that $Pu = 0$ and such that P is non-characteristic with respect to $y_0 = 0$. Concretely, this means that the coefficient of ∂_0^d is constant modulo y_0. As $y_0 u = 0$, we can therefore assume without loss of generality that ∂_0^d appear with coefficient 1 in P. Let us choose P in such a way that d is minimal. The commutator $[y_0, P]$ contains the term $-d\partial_0^{d-1}$, and since

$$[y_0, P]u = y_0 P u - P y_0 u = 0,$$

we conclude by minimality that $d = 0$, and hence that $u = 0$. This proves (b).

To prove the other two assertions, we choose a good filtration $F_\bullet M$, with $\operatorname{gr}^F M$ finitely generated over $\operatorname{gr}^F D(B) = B[\partial_0, \dots, \partial_n]$. Set $\mathcal{N} = f^* \mathcal{M}$ and $N = \Gamma(X, \mathcal{N})$, so that

$$N = A \otimes_B M.$$

This time, tensoring with A is no longer an exact functor, but we can still define a filtration on N by setting

$$F_j N = \operatorname{im}\bigl(A \otimes_B F_j M \to A \otimes_B M\bigr).$$

With this definition, each $\operatorname{gr}_j^F N$ is a quotient of $B \otimes_A \operatorname{gr}_j^F M$, and by exactly the same calculation as before, the $A[\partial_1, \dots, \partial_n]$-module $\operatorname{gr}^F N$ is a quotient of $A \otimes_B \operatorname{gr}^F M$, considered as an $A[\partial_1, \dots, \partial_n]$-module through the morphism in (16.7).

Now I claim that $A \otimes_B \operatorname{gr}^F M$ is finitely generated over $A[\partial_1, \dots, \partial_n]$. Indeed, $\operatorname{gr}^F M$ is finitely generated over $B[\partial_0, \dots, \partial_n]$ (because $F_\bullet M$ is good), and so $A \otimes_B \operatorname{gr}^F M$ is finitely generated over $A[\partial_0, \dots, \partial_n]$. By the non-characteristic property, the support inside $\operatorname{Spec} A[\partial_0, \dots, \partial_n]$ is finite over $\operatorname{Spec} A[\partial_1, \dots, \partial_n]$, and so the claim follows from Lemma 16.9. Therefore $\operatorname{gr}^F N$, which is a quotient, is also finitely generated over $A[\partial_1, \dots, \partial_n]$, proving that $\mathcal{N} = f^* \mathcal{M}$ is coherent over $\mathscr{D}_X$. This argument also shows that

$$\operatorname{Ch}(\mathcal{N}) \subseteq df\bigl(p_2^{-1} \operatorname{Ch}(\mathcal{M})\bigr),$$

because the support of $A \otimes_B \operatorname{gr}^F M$ contains the support of the quotient module $\operatorname{gr}^F N$. Some extra work is required to show that the two sides are actually equal. (In brief, one has to construct a good filtration $F_\bullet M$ such that $\operatorname{gr}_j^F N = A \otimes_B \operatorname{gr}_j^F M$.)

Exercises.

Exercise 16.1. Suppose that $X \subseteq \mathbb{A}^n$ is a nonsingular subvariety. Determine the set of hyperplanes $H \subseteq \mathbb{A}^n$ such that $p_2^{-1}(T_X^* \mathbb{A}^n)$ is finite over its image in $T^* H$.

$$\begin{tikzcd} H \times_{\mathbb{A}^n} T^* \mathbb{A}^n \arrow[r] \arrow[d, "p_2"] & T^* H \\ T^* \mathbb{A}^n & \end{tikzcd}$$