Lecture 15 (March 25)

Group schemes. This was the first lecture after spring break, so I briefly reviewed what we had done before the break. Let X be an abelian variety over an algebraically closed field k. We are interested in the group

$$\operatorname{Pic}^0(X) = \left\{ L \in \operatorname{Pic}(X) \;\middle|\; t_x^* L \cong L \text{ for every } x \in X \right\}$$

of translation-invariant line bundles on X, and in particular, in constructing an abelian variety $\hat{X}$, the so-called "dual" abelian variety, that is isomorphic to $\operatorname{Pic}^0(X)$ as a group. Let L be an ample line bundle. We showed that

$$\varphi_L \colon X \to \operatorname{Pic}^0(X), \quad \varphi_L(x) = t_x^* L \otimes L^{-1},$$

is a surjective homomorphism, and that $K(L) = \ker \varphi_L$ is a finite subgroup of X. In characteristic zero, we then defined the dual abelian variety as the quotient

$$\hat{X} = X/K(L).$$

We also constructed a universal line bundle P on $X \times \hat{X}$, called the Poincaré bundle, with the property that

$$(\operatorname{id} \times \pi)^* P \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}.$$

The pair $(\hat{X}, P)$ serves as a "moduli space" for families of line bundles in $\operatorname{Pic}^0(X)$, but we were only able to prove this for families parametrized by normal varieties.

To construct the dual abelian variety in all characteristics, we need to take into account that $K(L)$ is not just a finite set, but that it has a natural scheme structure. This follows from the scheme version of the seesaw theorem. Indeed, by Proposition 14.8, there is a maximal closed subscheme $X_0 \subseteq X$ with the property that the line bundle

$$M = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

is trivial on $X \times X_0$; the set of closed points of X_0 is exactly our subgroup $K(L)$, and this endows $K(L)$ with a scheme structure. In fact, $K(L)$ is an example of a "group scheme": a group-object in the category of schemes (of finite type over k).

Definition 15.1. A *group scheme* is a scheme G (of finite type over the field k) with a closed point $e \in G(k)$ and two morphisms

$$m \colon G \times G \to G \quad \text{and} \quad i \colon G \to G,$$

subject to the following conditions:

(1) m is associative, meaning that the diagram

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

is commutative.

(2) e is the unit element, meaning that the diagram

$$\begin{tikzcd}
G \arrow[dr, "{(\mathrm{id},e)}"'] \arrow[drr, bend left, "\mathrm{id}"] & & \\
& G \times G \arrow[r, "m"] & G \\
G \arrow[ur, "{(e,\mathrm{id})}"] \arrow[urr, bend right, "\mathrm{id}"'] & &
\end{tikzcd}$$

is commutative; here we view e as a morphism $e \colon \operatorname{Spec} k \to G$.

(3) i is the inverse, meaning that the diagram

$$\begin{tikzcd}
 & G \times G \arrow[dr, bend left, "m"] & \\
G \arrow[ur, bend left, "{(\mathrm{id},i)}"] \arrow[r] \arrow[dr, bend right, "{(i,\mathrm{id})}"'] & \operatorname{Spec} k \arrow[r, "e"] & G \\
 & G \times G \arrow[ur, bend right, "m"'] &
\end{tikzcd}$$

is commutative.

If G is a group scheme, then for every scheme S (of finite type over k), the set of G-valued points $G(S) = \operatorname{Hom}_{\operatorname{Spec} k}(S, G)$ becomes a group; conversely, if $G(S)$ is a group in a way that is functorial in S, then G has the structure of a group scheme.

Example 15.2. An abelian variety is obviously a group scheme; by definition, every abelian variety is reduced and irreducible. If we define

$$X_n = \ker\bigl(n_X \colon X \to X\bigr)$$

as the kernel of the morphism $x \mapsto nx$, then X_n is a closed subscheme of length $\deg n_X = n^{2 \dim X}$. We saw earlier that it has $n^{2 \dim X}$ points when n is not a multiple of the characteristic $\operatorname{char}(k)$; but for example X_p always has at most $p^{\dim X}$ many points, and must therefore be nonreduced.

Example 15.3. For $n \geq 1$, the n-th roots of unity form a group scheme

$$\mu_n = \operatorname{Spec} k[x]/(x^n - 1).$$

The group operation is given by the morphism of k-algebras

$$k[x]/(x^n - 1) \to k[y, z]/(y^n - 1, z^n - 1), \quad x \mapsto yz.$$

When the field k has characteristic p, the group scheme μ_p is nonreduced and only has a single closed point, because

$$k[x]/(x^p - 1) = k[x]/(x - 1)^p.$$

Example 15.4. In characteristic p, the Frobenius morphism

$$F \colon k[x] \to k[x], \quad F(x) = x^p,$$

is a ring homomorphism. The fiber over the origin is the group scheme

$$\operatorname{Spec} k[x]/(x^p),$$

which is again nonreduced with a single closed point. (The group operation is now $x \mapsto y + z$, in the same notation as in the previous example.)

The examples show that, in characteristic p, group schemes can have a nontrivial (meaning nonreduced) scheme structure. In characteristic zero, this does not happen, because of the following theorem.

Theorem 15.5. *Every group scheme over a field of characteristic* 0 *is nonsingular.*

The proof has two steps. First, one shows that the sheaf of Kähler differentials $\Omega^1_{G/k}$ on a group scheme G is always locally free. Recall that, according to one construction of the Kähler differentials, $\Omega^1_{G/k} \cong \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of the diagonal $\Delta \colon G \to G \times G$. Because G is a group scheme, we can describe the diagonal in terms of the group operations. Let

$$s = m \circ (\mathrm{id}, i) \colon G \times G \to G$$

be the morphism that acts on closed points as $s(x,y) = xy^{-1}$. One can show that

$$\begin{array}{ccc} G & \xrightarrow{\Delta} & G \times G \\ \big\downarrow & & \big\downarrow{\scriptstyle s} \\ \operatorname{Spec} k & \xrightarrow{e} & G \end{array}$$

is a Cartesian diagram, and therefore

$$\Omega^1_{G/k} \cong \mathcal{I}/\mathcal{I}^2 \cong s^*\big(\mathfrak{m}_e/\mathfrak{m}_e^2\big) \cong \mathscr{O}_G \otimes_k \mathfrak{m}_e/\mathfrak{m}_e^2,$$

where $\mathfrak{m}_e$ is the ideal sheaf of the closed point $e \in G(k)$. This proves that the sheaf of Kähler differentials is locally free of rank equal to the dimension of the k-vector space $\mathfrak{m}_e/\mathfrak{m}_e^2$. This much is true independently of the characteristic of the field.

Example 15.6. For $\mu_p = \operatorname{Spec} k[x]/(x^p - 1)$, we have $d(x-1)^p = p(x-1)^{p-1} = 0$, and so the module of Kähler differentials is isomorphic to

$$\Omega^1_{k[x]/k} \otimes_{k[x]} k[x]/(x^p - 1) \cong k[x]/(x^p - 1),$$

hence free of rank one.

The characteristic zero magic happens in the following lemma.

Lemma 15.7. *Let X be a scheme of finite type over a field k of characteristic zero. If the sheaf of Kähler differentials $\Omega^1_{X/k}$ is locally free, then X is nonsingular.*

Proof. I did not present the proof in class, but here it is. Let $x \in X(k)$ be an arbitrary closed point. It is enough to show that the local ring $\mathscr{O}_{X,x}$ is regular. So we may assume that $(A, \mathfrak{m})$ is a local k-algebra with residue field $A/\mathfrak{m} \cong k$, and that the module of Kähler differentials $\Omega^1_{A/k}$ is locally free. We need to show that A is regular, which means that $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$. Set $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$, and choose n elements $f_1, \dots, f_n \in \mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis over k. Because $\Omega^1_{A/k} \otimes_A k \cong \mathfrak{m}/\mathfrak{m}^2$, the rank of the free A-module $\Omega^1_{A/k}$ is equal to n, and so we have an isomorphism of A-modules

$$\Omega^1_{A/k} \cong A^n.$$

By the universal property of the Kähler differentials, this gives us n derivations $\delta_1, \dots, \delta_n \in \operatorname{Der}_k(A)$, with the property that $\delta_i(f_j) = 1$ if $i = j$, and 0 otherwise. It follows that $\delta_i(\mathfrak{m}^\ell) \subseteq \mathfrak{m}^{\ell-1}$ for all $\ell \geq 1$.

Now both $\dim_k \mathfrak{m}/\mathfrak{m}^2$ and $\dim A$ don't change under completion, and so we may replace A by its completion

$$\hat{A} = \varprojlim_\ell A/\mathfrak{m}^\ell.$$

Because $\delta_i(\mathfrak{m}^\ell) \subseteq \mathfrak{m}^{\ell-1}$, our derivations extend to $\hat{A}$ as well; we may therefore assume that A is complete to begin with. Because A is complete, we then get a homomorphism of k-algebras

$$\alpha \colon k[[x_1, \dots, x_n]] \to A, \quad \alpha(x_i) = f_i,$$

from the ring of formal power series, and α is easily seen to be surjective. For any $f \in A$, we denote by $f(0) \in k$ its image in $A/\mathfrak{m}$. Because $\operatorname{char}(k) = 0$, we can also define a function

$$\beta \colon A \to k[[x_1, \dots, x_n]], \quad \beta(f) = \sum_{k_1, \dots, k_n} \frac{1}{k_1! \cdots k_n!} \big(\delta_1^{k_1} \cdots \delta_n^{k_n} f\big)(0),$$

that sends every $f \in A$ to its Taylor series; a short computation proves that β is a ring homomorphism. The composition

$$\beta \circ \alpha \colon k[[x_1, \dots, x_n]] \to k[[x_1, \dots, x_n]]$$

is the identity modulo $(x_1, \dots, x_n)^2$, and is therefore an automorphism; in particular, α must be injective, and so α is an isomorphism. This proves that A is a regular local ring. ☐

We also need talk briefly about quotients. Suppose that G is a finite (hence affine) group scheme. An action of G on a scheme X is a morphism $G \times X \to X$, subject to the condition that certain diagrams commute. As in Lecture 12, one can define the quotient X/G, under the assumption that the orbit of every closed point is contained in an affine open subset of X. If $U = \operatorname{Spec} A$ is a G-invariant affine open subset, then $G \times U \to U$ corresponds to a morphism of k-algebras

$$\delta \colon A \to \Gamma(G, \mathscr{O}_G) \otimes_k A,$$

and one defines the subalgebra of G-invariant functions as

$$A^G = \left\{ f \in A \;\middle|\; \delta(f) = 1 \otimes f \right\}.$$

One can show that A^G is again a finitely-generated k-algebra, and the quotient U/G is defined as $\operatorname{Spec} A^G$.

The dual abelian variety in general. Let X be an abelian variety, and let L be an ample line bundle on X. Consider the line bundle

$$M = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \tag{15.8}$$

on $X \times X$, and let $K(L) \subseteq X$ be the maximal closed subscheme such that the restriction of M to $X \times K(L)$ is trivial; the set of closed points of this subscheme is the group $\ker \varphi_L$ that we used earlier. This is actually a group scheme: the group operation is induced by $m \colon X \times X \to X$.

Lemma 15.9. *The group operation $m \colon X \times X \to X$ on the abelian variety restricts to a morphism $m \colon K(L) \times K(L) \to K(L)$.*

Proof. Set $K = K(L)$. We need to show that the composition

$$K \times K \longrightarrow X \times X \xrightarrow{m} X$$

factors through the closed subscheme K. By the universal property in Proposition 14.8, this is equivalent to the pullback line bundle $(\mathrm{id} \times m)^* M$ being trivial on $X \times K \times K$. We are going to use the following notation:

$$\begin{tikzcd}
X & & X \times K \arrow[ll, "p_1"'] \arrow[r, "m"] & X & \\
X \times K \arrow[u, "m"] & X \times K \times K \arrow[l, "p_{12}"'] \arrow[ur, phantom] & & & \\
\end{tikzcd}$$

$$\begin{tikzcd}
X & X \times K \arrow[l, "p_1"'] \arrow[r, "m"] & X & \\
X \times K \arrow[u, "m"] & X \times K \times K \arrow[l, "p_{12}"'] \arrow[u, "m \times \mathrm{id}"'] \arrow[r, "\mathrm{id} \times m"] \arrow[d, "p_{23}"] & X \times X \arrow[u, "m"'] \arrow[r, "p_1"] \arrow[d, "p_2"] & X \\
& K \times K \arrow[r, "m"] & X &
\end{tikzcd}$$

Because of (15.8), we have

$$(\mathrm{id} \times m)^* M \cong (m \times \mathrm{id})^* m^* L \otimes p_{23}^* m^* L^{-1} \otimes p_1^* L^{-1}.$$

We can rewrite the first factor as

$$(m \times \mathrm{id})^* m^* L \cong (m \times \mathrm{id})^* M \otimes (m \times \mathrm{id})^* p_1^* L \otimes p_3^* L \cong p_{12}^* m^* L \otimes p_3^* L,$$

because M is trivial on $X \times K$ (by definition of K). Similarly, we have

$$p_{12}^* m^* L \cong p_{12}^* M \otimes p_1^* L \otimes p_2^* L \cong p_1^* L \otimes p_2^* L,$$

again because M is trivial on $X \times K$. Combining the three previous lines gives

$$\begin{aligned}
(\mathrm{id} \times m)^* M &\cong p_1^* L \otimes p_2^* L \otimes p_3^* L \otimes p_{23}^* m^* L^{-1} \otimes p_1^* L^{-1} \\
&\cong p_{23}^* \bigl(m^* L^{-1} \otimes p_1^* L \otimes p_2^* L \bigr) \cong p_{23}^* M^{-1},
\end{aligned}$$

which is trivial because M is trivial on $K \times K$. This proves the lemma. □

We can now define the dual abelian variety as the quotient

$$\hat{X} = X/K(L)$$

by the finite group scheme $K(L)$. If we let $\pi \colon X \to \hat{X}$ be the quotient morphism, we again get a Poincaré bundle P on the product $X \times \hat{X}$, with the property that

$$(\mathrm{id} \times \pi)^* P \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}.$$

By construction, the two line bundles

$$P|_{X \times \{0\}} \quad \text{and} \quad P|_{\{0\} \times \hat{X}}$$

are trivial, and the maximal closed subscheme $Z \subseteq \hat{X}$ such that P is trivial on $X \times Z$ is the reduced singleton $Z = \{0\}$. The pair $(\hat{X}, P)$ is now a moduli space for translation-invariant line bundles on X on the category of all schemes (of finite type over k). Indeed, one can prove the following universal property (similar to what we did in Lecture 13, but using Proposition 14.8).

Theorem 15.10. *Given a scheme S, and a line bundle L on $X \times S$ such that*

$$L_s = L|_{X \times \{s\}} \in \mathrm{Pic}^0(X)$$

for every closed point $s \in S(k)$, and such that $L|_{\{0\} \times S}$ is trivial, there is a unique morphism $f \colon S \to \hat{X}$ with the property that $L \cong (\mathrm{id} \times f)^* P$.

Cohomology of the structure sheaf. We showed that nontrivial line bundles in $\mathrm{Pic}^0(X)$ have no cohomology whatsoever. But we still haven't computed the cohomology groups of the trivial line bundle $\mathscr{O}_X$. We are going to do this by computing the cohomology of the Poincaré bundle P at the same time. The result is exactly the same as over the complex numbers.

Theorem 15.11. *Let X be an abelian variety of dimension g.*

(a) *We have* $\dim_k H^i(X, \mathscr{O}_X) = \binom{g}{i}$.

(b) *The cohomology of the Poincaré bundle is*

$$H^i(X \times \hat{X}, P) \cong \begin{cases} 0 & \text{if } i \neq g, \\ k & \text{if } i = g. \end{cases}$$

From (a), it follows that the natural map

$$\bigwedge^i H^1(X, \mathscr{O}_X) \to H^i(X, \mathscr{O}_X)$$

is an isomorphism of k-vector spaces. We will carry out the proof of the theorem in six steps; the main ingredient is (as usual) the base change theorem.

Step 1. Let $p_2 \colon X \times \hat{X} \to \hat{X}$ be the second projection. For $i \in \mathbb{N}$, define

$$\mathscr{F}_i = R^i (p_2)_* P,$$

which is a coherent sheaf on $\hat{X}$. Because $\dim X = g$, we have $\mathscr{F}_i = 0$ for $i > g$. We are going to prove (b) by computing these higher direct image sheaves. For any closed point $\alpha \in \hat{X}(k)$, we set

$$P_\alpha = P|_{X \times \{\alpha\}} \in \mathrm{Pic}^0(X),$$

and observe that P_α is trivial iff $\alpha = 0$. By Observation 7 in Lecture 13, we therefore have $H^i(X, P_\alpha) = 0$ for all $i \in \mathbb{Z}$ and all $\alpha \neq 0$; the base change theorem (in Corollary 9.9) therefore tells us that $\mathscr{F}_i$ is supported at the closed point $0 \in \hat{X}(k)$.

Step 2. Since $\operatorname{Supp} \mathscr{F}_i = \{0\}$, we have $H^j(\hat{X}, \mathscr{F}_i) = 0$ for $j > 0$, and so we get (from the Leray spectral sequence) that

$$H^i\big(X \times \hat{X}, P\big) \cong H^0\big(\hat{X}, \mathscr{F}_i\big) = 0$$

for $i > g$. From this, one can deduce by Serre duality that the same thing is true for $i < g$, and hence that $\mathscr{F}_i = 0$ for $i \neq g$; this is a nice exericse, but we will prove it in a different way in the next two steps of the argument.

Step 3. Now let's study what happens near the point $0 \in \hat{X}(k)$. We can work over the local ring $A = \mathscr{O}_{\hat{X},0}$; as usual, we denote the maximal ideal by $\mathfrak{m} = \mathfrak{m}_0$. According to Theorem 9.4, we can find a bounded complex

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely-generated free A-modules that "universally" computes the higher direct images $R^i(p_2)_*P$, in the sense that for any A-algebra B, one has

$$H^i\big(K^\bullet \otimes_A B\big) \cong H^i\big(X \times \hat{X}, P \otimes_A B\big).$$

Since we are working over a local ring, we may choose the complex $K^\bullet$ to be minimal, which means that all the differentials $d\colon K^i \to K^{i+1}$ have entries in the maximal ideal $\mathfrak{m}$. From

$$H^i\big(K^\bullet \otimes_A k\big) \cong H^i\big(X, P_0\big) \cong H^i(X, \mathscr{O}_X)$$

and minimality, we see that

$$\operatorname{rk} K^i = \dim_k H^i(X, \mathscr{O}_X).$$

In particular, we have $K^i = 0$ for $i > g$, and so our minimal complex takes the form

$$0 \to K^0 \to K^1 \to \cdots \to K^g \to 0.$$

Let M_i be the finitely-generated A-module corresponding to $\mathscr{F}_i = R^i(p_2)_*P$; these are the cohomology modules of the complex $K^\bullet$.

Step 4. The following simple lemma from commutative algebra now lets us conclude that $M_i = 0$ for $i < g$.

Lemma 15.12. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension g. Let*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

be a bounded complex of finitely-generated free A-modules, such that all cohomology modules $H^i(K^\bullet)$ have finite length. Then $H^i(K^\bullet) = 0$ for $i < g$.

Proof. The statement is trivial for $g = 0$, and so we can argue by induction on $g \geq 0$. Choose an element $f \in \mathfrak{m}$ such that $f \notin \mathfrak{m}^2$; then the quotient ring $\bar{A} = A/Af$ is regular of dimension $g - 1$ (by dimension theory). If we set $\bar{K}^\bullet = K^\bullet \otimes_A \bar{A}$, we get a short exact sequence of complexes

$$0 \longrightarrow K^\bullet \xrightarrow{f} K^\bullet \longrightarrow \bar{K}^\bullet \longrightarrow 0$$

and therefore an exact sequence in cohomology

$$H^i(K^\bullet) \xrightarrow{f} H^i(K^\bullet) \longrightarrow H^i(\bar{K}^\bullet) \longrightarrow H^{i+1}(K^\bullet) \xrightarrow{f} H^{i+1}(K^\bullet).$$

Because all cohomology modules of $K^\bullet$ have finite length, it follows that the cohomology modules of $\bar{K}^\bullet$ also have finite length. By induction, we therefore get $H^i(\bar{K}^\bullet) = 0$ for $i < g - 1$. From the exact sequence, $f\colon H^i(K) \to H^i(K)$ is then injective for $i < g$; but because $H^i(K)$ has finite length, it is annihilated by some power of f, and so $H^i(K) = 0$ for $i < g$. $\square$

If we apply this to our complex, we find that $M_i = 0$ for $i < g$, and hence that M_g is the only nontrivial cohomology module of $K^\bullet$. In other words, $K^\bullet$ is a minimal free resolution of the A-module M_g.

Step 5. Now let's combine this with what we know about the Poincaré bundle (from the construction of $\hat{X}$). We have $H^0(X, \mathscr{O}_X) = k$, and so $K^0 \cong A$; setting $n = \operatorname{rk} K^1$, we also get $K^1 \cong A^n$. The differential

$$d\colon K^0 \to K^1$$

is therefore given by n elements $f_1, \dots, f_n \in \mathfrak{m}$ (by minimality). We showed during the proof of Proposition 14.8 that the maximal closed subscheme of $\hat{X}$ over which P is trivial is defined by the ideal $(f_1, \dots, f_n)$. In our case, this closed subscheme is $\{0\}$, and so we must have $(f_1, \dots, f_n) = \mathfrak{m}$. Consider now the dual complex

$$0 \to (K^n)^* \to \dots \to (K^1)^* \to (K^0)^* \to 0.$$

By the lemma from Step 4, this complex is again exact in all places except at the right end, and there, the cohomology is $A/(f_1, \dots, f_n) = A/\mathfrak{m} = k$. The dual complex is therefore a minimal free resolution of the residue field k.

Step 6. But we know from commutative algebra what the minimal free resolution of $A/\mathfrak{m}$ looks like in a regular local ring: it is the Koszul complex for a regular sequence $x_1, \dots, x_g \in \mathfrak{m}$. The Koszul complex is the tensor product of the g complexes

$$0 \longrightarrow A \xrightarrow{x_i} A \longrightarrow 0$$

and therefore has the shape

$$0 \to A^{\binom{g}{0}} \to A^{\binom{g}{1}} \to A^{\binom{g}{2}} \to \dots \to A^{\binom{g}{g-1}} \to A^{\binom{g}{g}} \to 0.$$

Because minimal free resolutions are unique (up to isomorphism), the dual complex of $K^\bullet$, and hence $K^\bullet$ itself, must be a Koszul complex as well. This gives

$$\dim_k H^i(X, \mathscr{O}_X) = \operatorname{rk} K^i = \binom{g}{i},$$

which proves (a). We also find that $M_g = H^g(K^\bullet) \cong A/\mathfrak{m}$, and so $\mathscr{F}_g$ is the structure sheaf of the closed point $0 \in \hat{X}(k)$. We now get (b) from the computation in Step 2. □

Note. In fact, we have shown that

$$R^i(p_2)_* P \cong \begin{cases} 0 & \text{if } i \neq g, \\ \mathscr{O}_0 & \text{if } i = g. \end{cases}$$

This result will be important when we study derived categories of abelian varieties.

Corollary 15.13. *We have* $\dim H^q(X, \Omega^p_{X/k}) = \binom{g}{p}\binom{g}{q}$.

Proof. The Kähler differentials $\Omega^1_{X/k}$ are locally free of rank $g = \dim X$. Therefore $\Omega^p_{X/k} = \bigwedge^p \Omega^1_{X/k}$ is locally free of rank $\binom{g}{p}$, and the formula follows from the theorem. □