Lecture 14 (March13)

Properties of the dual abelian variety. Last time, we constructed the dual abelian variety $\hat{X}$ and the Poincaré bundle P on $X \times \hat{X}$. For a point $\alpha \in \hat{X}$, we introduced the notation

$$P_\alpha = P|_{X \times \{a\}} \in \operatorname{Pic}^0(X);$$

this is the line bundle corresponding to α under the isomorphism $\hat{X} \cong \operatorname{Pic}^0(X)$. In class, I first went over the proof of the universal property again. During the proof, we used the fact that the field k has characteristic zero; the general case needs a bit more work.

We then looked at a few basic properties of the construction. First, let L be any line bundle on the abelian variety X, and consider the homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X), \quad \phi_L(x) = t_x^* L \otimes L^{-1}.$$

This is in fact a morphism of abelian varieties; more precisely, under our isomorphism $\hat{X} \cong \operatorname{Pic}^0(X)$, the homomorphism ϕ_L comes from a morphism $f \colon X \to \hat{X}$. For the proof, consider the line bundle

$$K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

on the product $X \times X$. We have

$$K|_{X \times \{x\}} \cong t_x^* L \otimes L^{-1} \quad \text{and} \quad K|_{\{0\} \times X} \cong \mathscr{O}_X,$$

and so we can apply the universal property (which we called (B) last time). This gives us a unique morphism $f \colon X \to \hat{X}$ such that $K \cong (\operatorname{id} \times f)^* P$. Restricting to $X \times \{x\}$, we get $P_{f(x)} \cong t_x^* L \otimes L^{-1} = \phi_L(x)$, and so f does indeed realize ϕ_L. Note that f is a group homomorphism (because ϕ_L is).

The next result says that the dual abelian variety is really a functor on the category of abelian varieties. Recall that a morphism of abelian varieties is a morphism that is also a group homomorphism. We showed that any morphism $f \colon X \to Y$ with $f(0) = 0$ is a homomorphism.

Proposition 14.1. *Let $f \colon X \to Y$ be a morphism of abelian varieties. Then the pullback homomorphism $f^* \colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ defines a morphism $\hat{f} \colon \hat{Y} \to \hat{X}$.*

Proof. Let's write P_X for the Poincaré bundle on $X \times \hat{X}$, and P_Y for the one on $Y \times \hat{Y}$. On $X \times \hat{Y}$, consider the line bundle $(f \times \operatorname{id})^* P_Y$. Its restriction to $\{0\} \times \hat{Y}$ is trivial because $f(0) = 0$; the restrictions to $X \times \{\alpha\}$ are in $\operatorname{Pic}^0(X)$ by Observation 6 from last time (because this holds when $\alpha = 0$). By the universal property for $\hat{X}$, there is thus a unique morphism $\hat{f} \colon \hat{Y} \to \hat{X}$ such that

$$(f \times \operatorname{id})^* P_Y \cong (\operatorname{id} \times \hat{f})^* P_X. \tag{14.2}$$

Here is a diagram of the two morphisms:

$$\begin{tikzcd} X \times \hat{Y} \arrow[r, "f \times \operatorname{id}"] \arrow[d, "\operatorname{id} \times \hat{f}"] & Y \times \hat{Y} \\ X \times \hat{X} \end{tikzcd}$$

If we restrict the isomorphism to $X \times \{\alpha\}$, we obtain

$$P_{X, \hat{f}(\alpha)} \cong f^* P_{Y, \alpha},$$

which is saying that the morphism $\hat{f}$ realizes the pullback f^* on line bundles. □

We can say a bit more in the case of isogenies.

Proposition 14.3. *Let $f\colon X \to Y$ be an isogeny. Then $\hat{f}\colon \hat{Y} \to \hat{X}$ is also an isogeny, and $\ker f$ and $\ker \hat{f}$ are dual abelian groups, in the sense that*

$$\ker \hat{f} \cong \operatorname{Hom}\big(\ker f, k^\times\big).$$

Proof. We showed at the end of Lecture 12 that

$$\ker\big(f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)\big) \cong \operatorname{Hom}\big(\ker f, k^\times\big)$$

is true for separable isogenies (and all isogenies are separable because we are assuming that k has characteristic zero). So it suffices to show that if f^*L is trivial for a line bundle $L \in \operatorname{Pic}(Y)$, then $L \in \operatorname{Pic}^0(Y)$. This implies that $\ker \hat{f}$ is dual to $\ker f$, hence finite, and then $\hat{f}$ must be an isogeny for dimension reasons. The proof is very easy: $\ker f^*$ is a finite group (because it is dual to the finite group $\ker f$), and so L has finite order; but we showed that any line bundle of finite order is in $\operatorname{Pic}^0(Y)$. □

Example 14.4. The isogeny $n_X\colon X \to X$ has the property that $\hat{n}_X\colon \hat{X} \to \hat{X}$ is equal to $n_{\hat{X}}$. This follows from the identity $n_X^* L \cong L^n$ for $L \in \operatorname{Pic}^0(X)$ that we proved last time.

Example 14.5. Over the complex numbers, we can write an abelian variety as $X = V/\Gamma$, where V is a g-dimensional complex vector space, and Γ is a lattice of rank $2g$. The dual abelian variety is

$$\operatorname{Pic}^0(X) = H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z}).$$

Now $H_1(X,\mathbb{Z}) \cong \Gamma$, and therefore

$$H^1(X, \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$$

is the lattice dual to Γ. We also have

$$H^1(X, \mathscr{O}_X) \cong \operatorname{Hom}_{\mathbb{C}}\big(\bar{V}, \mathbb{C}\big),$$

with a conjugate-linear functional $f\colon V \to \mathbb{C}$ mapping to the translation-invariant $(0,1)$-form df. The embedding of the dual lattice works by extending a homomorphism $\varphi\colon \Gamma \to \mathbb{Z}$ uniquely to a linear functional $\varphi_{\mathbb{C}}\colon \Gamma \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}$, and then projecting to the second summand in

$$\operatorname{Hom}_{\mathbb{C}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{C}) \cong \operatorname{Hom}_{\mathbb{C}}(V \oplus \bar{V}, \mathbb{C}) \cong \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}) \oplus \operatorname{Hom}_{\mathbb{C}}(\bar{V}, \mathbb{C}).$$

This explains the reason for calling $\operatorname{Pic}^0(X)$ the “dual” abelian variety.

Symmetric description of the dual abelian variety. While this is not clear from our construction of $\hat{X}$ (as a quotient of X), the two abelian varieties X and $\hat{X}$ really play the same role. To make this precise, we make the following definition.

Definition 14.6. A *divisorial correspondence* between two abelian varieties X and Y is a line bundle Q on $X \times Y$ such that $Q|_{\{0\}\times Y}$ and $Q|_{X\times\{0\}}$ are trivial.

We could realize Q by a divisor on $X \times Y$, which would then be a divsorial correspondence in the proper sense, but it is much better to work with line bundles. By Observation 6 from last time, we have

$$Q|_{\{x\}\times Y} \in \operatorname{Pic}^0(Y) \quad \text{and} \quad Q|_{X\times\{y\}} \in \operatorname{Pic}^0(X)$$

for every $x \in X$ and every $y \in Y$.

Proposition 14.7. *Let X and Y be abelian varieties of the same dimension, and let Q be a divisorial correspondence between X and Y. Then the following two conditions are equivalent:*

(a) $Q|_{\{x\}\times Y}$ trivial implies that $x = 0$.
(b) $Q|_{X\times\{y\}}$ trivial implies that $y = 0$.

If either of these conditions is satisfied, then $X \cong \hat{Y}$ and $Y \cong \hat{X}$, and Q is isomorphic to the pullback of both Poincaré bundles P_X and P_Y.

Proof. We only need to prove that (a) implies (b); the converse follows by interchanging X and Y. Let's first consider Q as a family of line bundles on Y. By the universal property of the dual abelian variety, we get a unique morphism $f\colon X \to \hat{Y}$ such that $Q \cong (f \times \mathrm{id})^* s^* P_Y$, where $s\colon Y \times \hat{Y} \to \hat{Y} \times Y$ is the morphism $s(y, \eta) = (\eta, y)$ that swaps the two factors. But (a) tells us that

$$P_{Y,f(x)} \cong Q|_{\{x\}\times Y}$$

is trivial only when $x = 0$, and so $\ker f = \{0\}$. Therefore f is injective, hence bijective (because $\dim X = \dim Y$), hence an isomorphism (because $\mathrm{char}(k) = 0$).

We can also view Q as a family of line bundles on X, and so we also get a unique morphism $g\colon Y \to \hat{X}$ such that $Q \cong (\mathrm{id} \times g)^* P_X$.

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{id} \times g"] \arrow[d, "f\times\mathrm{id}"] & X \times \hat{X} \\ \hat{Y} \times Y \end{tikzcd}$$

In order to prove (b), we need to show that g is injective. Let $K \subseteq \ker g$ be any finite subgroup of g; we shall argue that $K = \{0\}$, which is enough to conclude that g is injective. Because $K \subseteq \ker g$, we get a factorization

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[rr, bend left, "g"] & Z \arrow[r, "\tilde{g}"] & \hat{X}, \end{tikzcd}$$

where $Z = Y/K$ is the quotient. If we set $L = (\mathrm{id} \times \tilde{g})^* P_X$, which is a line bundle on $X \times Z$, then $Q \cong (\mathrm{id} \times \pi)^* L$. Viewing L as a family of line bundles on Z, we get a third morphism $h\colon X \to \hat{Z}$, with the property that $L \cong (h \times \mathrm{id})^* s^* P_Z$. Let $\hat{\pi}\colon \hat{Z} \to \hat{Y}$ be the morphism dual to $\pi\colon Y \to Z$. According to (14.2), we have

$$(\pi \times \mathrm{id})^* P_Z \cong (\mathrm{id} \times \hat{\pi})^* P_Y.$$

If we combine this with the formulas for Q and L, we get

$$\begin{aligned} Q &\cong (\mathrm{id} \times \pi)^* (h \times \mathrm{id})^* s^* P_Z \cong (h \times \mathrm{id})^* s^* (\pi \times \mathrm{id})^* P_Z \\ &\cong (h \times \mathrm{id})^* s^* (\mathrm{id} \times \hat{\pi})^* P_Y \cong (h \times \mathrm{id})^* (\hat{\pi} \times \mathrm{id})^* s^* P_Y \\ &\cong ((\hat{\pi} \circ h) \times \mathrm{id})^* s^* P_Y. \end{aligned}$$

But Q is also isomorphic to $(f \times \mathrm{id})^* s^* P_Y$, and so the uniqueness of the morphism (in the universal property of the dual abelian variety) implies that $f = \hat{\pi} \circ h$. In other words, we found a factorization

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend left, "f"] & \hat{Z} \arrow[r, "\hat{\pi}"] & \hat{Y}. \end{tikzcd}$$

Now f is an isomorphism by (a), and so h must be injective. For dimension reasons, h is then an isomorphism, and so $\hat{\pi}$ is an isomorphism as well. By Proposition 14.3, the kernel of $\hat{\pi}$ is dual to $K = \ker \pi$. Therefore K is trivial, and so $g\colon Y \to \hat{X}$ is injective, as claimed. This proves (b). Along the way, we have shown that

$$f\colon X \to \hat{Y} \quad \text{and} \quad g\colon Y \to \hat{X}$$

are isomorphisms, and that $(\mathrm{id} \times g)^* P_X \cong Q \cong (f \times \mathrm{id})^* s^* P_Y$. $\square$

We can apply this to the Poincaré bundle P_X on the product $X \times \hat{X}$; this is a divisorial correspondence, and $P_\alpha = P_X|_{X \times \{\alpha\}}$ is trivial only when $\alpha = 0$. The proposition then tells us that the dual abelian variety of $\hat{X}$ is isomorphic to the original abelian variety X, and that the Poincaré bundle $P_{\hat{X}}$ is isomorphic to $s^* P_X$, where $s\colon X \times \hat{X} \to \hat{X} \times X$ again swaps the two factors.

Positive characteristic and schemes. In the construction of the dual abelian variety, we had to assume that k has characteristic zero to prove the universal property. Ultimately, it comes down to the fact that when we have a line bundle L on $X \times S$, we are treating the locus in S such that L_s is trivial as a set, instead of as a scheme. (This applies in particular to the subgroup $K(L)$ inside X.) That is also the reason for the (unsatisfying) assumption that the parameter space S in the universal property needs to be normal. To fix these problems, we first need to revisit the seesaw theorem and make it works for schemes.

Proposition 14.8. *Let X be a complete variety, S a scheme, and L a line bundle on $X \times S$. There is a unique closed subscheme $S_0 \subseteq S$ such that:*

(a) $L|_{X \times S_0} \cong p_2^* L_0$ for a line bundle L_0 on S_0.
(b) If $f\colon T \to S$ is a morphism of schemes such that $(\mathrm{id} \times f)^* L \cong p_2^* K$ for a line bundle K on T, then f factors through S_0.

Proof. The proof is basically the same as that of Theorem 9.10, we just need to pay a little bit more attention to the details. For a closed point $s \in S(k)$, let's put as usual $L_s = L|_{X \times \{s\}}$. We already know that the set of $s \in S(k)$ such that L_s is trivial is closed in the Zariski topology. All we need to do is to put a natural scheme structure on this set. The problem being local, we may fix a point $s \in S(k)$ such that L_s is trivial, and then replace S by an affine open neighborhood $\operatorname{Spec} A$ of the point s. According to Theorem 9.4, we can find a bounded complex

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely-generated free A-modules – we can make them free by shrinking S, if necessary – such that for every B-algebra A, one has

$$H^p\big(X \times_{\operatorname{Spec} A} \operatorname{Spec} B, L \otimes_A B\big) \cong H^p(K^\bullet \otimes_A B).$$

We may further assume that the complex is minimal at the point s; if we let $\mathfrak{m} \subseteq A$ denote the maximal ideal corresponding to $s \in S(k)$, then this means that the complex $K^\bullet \otimes_A A/\mathfrak{m}$ has trivial differentials. Because this complex computes the cohomology of $L \otimes_A A/\mathfrak{m} \cong L_s$, and because L_s is trivial, we get $H^0(X, L_s) \cong k$, and so K^0 must have rank one, hence $K^0 \cong A$. Likewise, $K^1 \cong A^r$ for some $r \geq 1$, and the differential $d\colon K^0 \to K^1$ is therefore represented by r elements $f_1, \dots, f_r \in A$. For the time being, let $I = (f_1, \dots, f_r) \subseteq A$ be the ideal generated by these elements. Taking $B = A/I$, we get

$$H^0\big(X \times_{\operatorname{Spec} A} \operatorname{Spec}(A/I), L \otimes_A A/I\big) \cong H^0(K^\bullet \otimes_A A/I) \cong A/I, \tag{14.9}$$

because $d\colon K^0 \otimes_A A/I \to K^1 \otimes_A A/I$ is of course trivial by construction. So the restriction of L to the closed subscheme $X \times_{\operatorname{Spec} A} \operatorname{Spec}(A/I)$ has a nontrivial global section. In fact, we get a line bundle L_0 on $\operatorname{Spec}(A/I)$, corresponding to the free A/I-module in (14.9), and the global section is really a morphism from $p_2^* L_0$ to the restriction of L.

As in Theorem 9.10, we now repeat this procedure for the line bundle L^{-1}; this gives us several additional elements $g_1, \dots, g_p \in A$, which we add to the ideal I. The desired closed subscheme is then $S_0 = \operatorname{Spec}(A/I)$. The reason is that both L and L^{-1} have a nontrivial global section on $X \times_S S_0$ (and so L_s is trivial for every closed point of S_0). The argument above gives us a line bundle L_0 on S_0, and an isomorphism $p_2^* L_0 \cong L|_{X \times S_0}$. This proves (a).

For (b), we may assume (by uniqueness) that $T = \operatorname{Spec} B$ is affine and that the line bundle K is trivial. The morphism $f \colon T \to S$ is given by a morphism of k-algebras $\varphi \colon A \to B$, and to show that f factors through S_0, we need to prove that $I \subseteq \ker \varphi$. Because $(\mathrm{id} \times f)^* L \cong p_2^* K$, we get

$$B \cong H^0\bigl(X \times_{\operatorname{Spec} A} \operatorname{Spec} B, L \otimes_A B\bigr) \cong H^0(K^\bullet \otimes_A B),$$

and because $K^0 \cong A$, this is only possible if the differential $d \colon K^0 \otimes_A B \to K^1 \otimes_A B$ is zero. But this means exactly that $\varphi(f_1) = \cdots = \varphi(f_r) = 0$. $\square$

As before, this improved version of the seesaw theorem implies the theorem of the cube for schemes.

Corollary 14.10. *Let L be a line bundle on $X \times Y \times S$, where X, Y are complete varieties, and S is a scheme. Suppose that there are points $x_0 \in X$, $y_0 \in Y$, and $s_0 \in S$ such that the three line bundles*

$$L|_{\{x_0\} \times Y \times S}, \quad L|_{X \times \{y_0\} \times S}, \quad L|_{X \times Y \times \{s_0\}}$$

are trivial. Then L is trivial.

With this result in hand, we can now construct the dual abelian variety in general. Let L be an ample line bundle on X. We proved that

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

is surjective, and that its kernel $K(L)$ is a finite group. The dual abelian variety should therefore still be the quotient of X by this subgroup, in a suitable sense.

We first observe that the closed subgroup $K(L) \subseteq X$ has a natural scheme structure on it. Indeed, if we take the line bundle

$$M = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

on $X \times X$, and consider the first copy of X as the parameter space, then Proposition 14.8 shows that there is a unique closed subscheme $X_0 \subseteq X$ such that

$$L|_{X_0 \times X} \cong p_1^* L_0$$

for some line bundle L_0 on X_0. Because $M|_{\{0\} \times X}$ is trivial, L_0 must be trivial, and so $X_0 \subseteq X$ is the maximal closed subscheme of X such that $L|_{X_0 \times X}$ is trivial. The set of closed points of X_0 is of course our subgroup $K(L)$, and so this puts a scheme structure on $K(L)$. From now on, we are going to denote this subscheme by the same symbol $K(L)$. We'll show next time that the group operation $m \colon X \times X \to X$ restricts to a morphism $K(L) \times K(L) \to K(L)$, and this makes $K(L)$ into a "group scheme". We can then define the dual abelian variety as

$$\hat{X} = X/K(L),$$

but where we now take the scheme structure on $K(L)$ into account when taking the quotient. (In characteristic zero, every group scheme is reduced; but in positive characteristic, $K(L)$ might be nonreduced, and then the quotient is different.)