Lecture 13 (March 11)

Translation-invariant line bundles. Let X be an abelian variety. Over the complex numbers, $\operatorname{Pic}^0(X)$ is the space of holomorphic line bundles with trivial first Chern class; this is again an abelian variety of the same dimension. Our goal today is to construct this abelian variety over any field of characteristic zero. We showed in Lecture 6 that all line bundles in $\operatorname{Pic}^0(X)$ are translation-invariant, in the sense that $t_x^* L \cong L$ for every $x \in X$. We use this property as the definition over other fields (where we don't have a good theory of first Chern classes in cohomology).

Definition 13.1. If X is an abelian variety, we define

$$\operatorname{Pic}^0(X) = \left\{ L \in \operatorname{Pic}(X) \;\middle|\; t_x^* L \cong L \text{ for all } x \in X \right\},$$

the group of (isomorphism classes of) translation-invariant line bundles.

In terms of the group homomorphism

$$\phi_L \colon X \to \operatorname{Pic}(X), \quad \phi_L(x) = t_x^* L \otimes L^{-1},$$

the subgroup $\operatorname{Pic}^0(X) \subseteq \operatorname{Pic}(X)$ consists of all those line bundles for which $\phi_L \equiv 0$. By the theorem of the square, we have

$$t_y^* \phi_L(x) = t_{x+y}^* L \otimes t_y^* L^{-1} \cong t_x^* L \otimes L^{-1}$$

and so $\phi_L(x) \in \operatorname{Pic}^0(X)$. Therefore

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

takes values in the subgroup $\operatorname{Pic}^0(X)$. We are going to construct an abelian variety $\hat{X}$ that is isomorphic to $\operatorname{Pic}^0(X)$ as a group (in a functorial way).

We begin with series of observations about translation-invariant line bundles.

Observation 1. We have $L \in \operatorname{Pic}^0(X)$ iff $m^* L \cong p_1^* L \otimes p_2^* L$ on $X \times X$. This is a consequence of the seesaw theorem. Indeed, the restriction of the line bundle $m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$ to the slice $X \times \{x\}$ is isomorphic to $t_x^* L \otimes L^{-1}$, and therefore trivial when $L \in \operatorname{Pic}^0(X)$. Because the line bundle is also trivial on $\{0\} \times X$, it must be trivial on $X \times X$ by Theorem 9.10.

Observation 2. If $f, g \colon S \to X$ are two morphisms from a variety (or scheme) S, then $(f+g)^* L \cong f^* L \otimes g^* L$. This follows from Observation 1 by pulling back along the morphism $(f, g) \colon S \to X \times X$.

Observation 3. Let $n_X \colon X \to X$ be the morphism $n_X(X) = n \cdot x$. By induction, the previous observation implies that $n_X^* L \cong L^n$. In particular, $(-1)_X^* L \cong L^{-1}$, and so L is anti-symmetric.

Observation 4. For every $L \in \operatorname{Pic}(X)$, we have $n_X^* L \otimes L^{-n^2} \in \operatorname{Pic}^0(X)$. By rewriting the identity in Corollary 11.4, we get

$$n_X^* L \otimes L^{-n^2} \cong \left(L \otimes (-1)_X^* L^{-1} \right)^{(n-n^2)/2},$$

and so it is enough to prove that $L \otimes (-1)_X^* L^{-1} \in \operatorname{Pic}^0(X)$. We compute

$$\begin{aligned}
t_y^* \left(L \otimes (-1)_X^* L^{-1} \right) &\cong t_y^* L \otimes (-1)_X^* t_{-y}^* L^{-1} \\
&\cong t_y^* L \otimes (-1)_X^* \left(t_{-y}^* L^{-1} \otimes L \right) \otimes (-1)_X^* L^{-1} \\
&\cong t_y^* L \otimes \left(t_{-y}^* L \otimes L^{-1} \right) \otimes (-1)_X^* L^{-1} \\
&\cong L^2 \otimes L^{-1} \otimes (-1)_X^* L^{-1} \cong L \otimes (-1)_X^* L^{-1},
\end{aligned}$$

where we used the fact htat $t_{-y}^* L \otimes L^{-1} \in \operatorname{Pic}^0(X)$ (and Observation 2) to go from the second to the third line; and the identity $t_y^* L \otimes t_{-y}^* L \cong L^2$ from the theorem of the square to go from the third to the fourth line.

Observation 5. If $L \in \mathrm{Pic}(X)$ has finite order, then $L \in \mathrm{Pic}^0(X)$. Indeed, if L^n is trivial for some $n \geq 1$, then one has

$$0 = \phi_{L^n}(x) = n\phi_L(x) = \phi_L(nx)$$

for every $x \in X$, and because X is divisible, this implies that $\phi_L \equiv 0$ and hence that $L \in \mathrm{Pic}^0(X)$.

Observation 6. Let S be a variety, and let L be a line bundle on $X \times S$; as usual, we think of this as a family of line bundles $L_s = L|_{X \times \{s\}}$ on X, parametrized by the variety S. Then for any two points $s_0, s_1 \in S$, one has $L_{s_1} \otimes L_{s_0}^{-1} \in \mathrm{Pic}^0(X)$. What this means is that the connected components of $\mathrm{Pic}(X)$ are copies of $\mathrm{Pic}^0(X)$, in the sense that an irreducible (hence connected) family of line bundles can only change L_{s_0} by line bundles in $\mathrm{Pic}^0(X)$.

Proof. After replacing L by $L \otimes p_1^* L_{s_0}^{-1}$, we may assume that L_{s_0} is trivial; then the claim is that $L_s \in \mathrm{Pic}^0(X)$ for all $s \in S$. The restriction of L to $\{0\} \times S$ is a line bundle on S, hence locally trivial; after replacing S by an open subset, we may therefore assume in addition that $L|_{\{0\} \times S}$ is trivial. In order to show that $L_s \in \mathrm{Pic}^0(X)$, it is enough to prove that $m^* L_s \otimes p_1^* L_s^{-1} \otimes p_2^* L_s^{-1}$ is trivial. To do that, we go to the product $X \times X \times S$, and consider the line bundle

$$M = \mu^* L \otimes p_{12}^* L^{-1} \otimes p_{13}^* L^{-1},$$

where $\mu \colon X \times X \times S \to X \times S$ is the morphism $\mu(x, y, s) = (x + y, s)$. The assumptions on L imply that M is trivial on $X \times X \times \{s_0\}$, on $\{0\} \times X \times S$, and on $X \times \{0\} \times S$. The theorem of the cube implies that M is trivial, and this gives the result we want after restricting to $X \times X \times \{s\}$. □

Observation 7. If $L \in \mathrm{Pic}^0(X)$ is nontrivial, then $H^i(X, L) = 0$ for every $i \in \mathbb{Z}$.

Proof. We prove this by induction on $i \geq 0$. Suppose that $s \in H^0(X, L)$ is a nontrivial global section. Then $(-1)_X^* s$ is a nontrivial global section of $(-1)_X^* L \cong L^{-1}$, and so $s \otimes (-1)_X^* s$ is a nontrivial global section of $L \otimes L^{-1} \cong \mathscr{O}_X$, hence a nonzero constant (because X is complete). But then the original section s cannot vanish anywhere, and so L is trivial, contrary to our initial assumption.

For $i > 0$, consider the composition

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[rr, bend left, "\mathrm{id}"] & X \times X \arrow[r, "m"] & X \end{tikzcd}$$

where $j(x) = (x, 0)$ and $m(x, y) = x + y$. It gives us a factorization

$$\begin{tikzcd} H^i(X, L) \arrow[r, "m^*"] \arrow[rr, bend left, "\mathrm{id}"] & H^i(X \times X, m^* L) \arrow[r, "j^*"] & H^i(X, L). \end{tikzcd}$$

From Observation 1, we know that $m^* L \cong p_1^* L \otimes p_2^* L$, and so

$$H^i(X \times X, m^* L) \cong \bigoplus_{p+q=i} H^p(X, L) \otimes H^q(X, L)$$

by the Künneth formula. But now all summands are trivial (by induction), and so $H^i(X \times X, m^* L) = 0$; the above factorization then gives $H^i(X, L) = 0$ as well. □

Observation 8. If L is an ample line bundle, the homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

is surjective. This is the key result for describing $\operatorname{Pic}^0(X)$.

Proof. Fix a translation-invariant line bundle $M \in \operatorname{Pic}^0(X)$. We need to find a point $x \in X$ such that $M \cong t_x^* L \otimes L^{-1}$. Suppose to the contrary that no such point exists. We'll derive a contradiction by looking at the line bundle

$$K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* (L^{-1} \otimes M^{-1})$$

on the product $X \times X$. We have

$$K|_{\{x\} \times X} \cong t_x^* L \otimes L^{-1} \otimes M^{-1},$$

and because $t_x^* L \otimes L^{-1}$ is not isomorphic to M, this line bundle is nontrivial, and therefore has no cohomology (by the previous observation). According to Corollary 9.9, applied to the first projection $p_1 \colon X \times X \to X$, it follows that $R^i (p_1)_* K = 0$ for every $i \in \mathbb{Z}$. By the Leray spectral sequence (or an exercise in Hartshorne), we now get

$$H^i(X \times X, K) = 0$$

for all $i \in \mathbb{Z}$.

Now let's consider the second projection $p_2 \colon X \times X \to X$. Here we have

$$K|_{X \times \{x\}} \cong t_x^* L \otimes L^{-1},$$

which is trivial exactly when x belongs to the subgroup $K(L) = \ker \phi_L$. Since L is ample, $K(L)$ is a finite group by Theorem 11.7. Therefore $K|_{X \times \{x\}}$ has no cohomology except when $x \in K(L)$. Another application of base change shows that the support of the coherent sheaves $R^q(p_2)_* K$ is contained in $K(L)$, and so $H^p(X, R^q(p_2)_* K) = 0$ for $p \geq 1$ for dimension reasons. The Leray spectral sequence therefore degenerates and gives us isomorphisms

$$0 = H^i(X \times X, K) \cong H^0\big(X, R^i(p_2)_* K\big).$$

It follows that $R^i(p_2)_* K = 0$, and hence (by Corollary 9.9) that $K_{X \times \{x\}}$ has no cohomology for every $x \in X$. But this is absurd because this bundle is isomorphic to $\mathscr{O}_X$ when $x = 0$, and $H^0(X, \mathscr{O}_X) = k$. $\square$

If we take L to be an ample line bundle – which exists because X is projective (by Corollary 11.9) – then the homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

is surjective, and its kernel is the finite subgroup $K(L)$. As a group, $\operatorname{Pic}^0(X)$ is therefore isomorphic to the quotient $X/K(L)$.

Example 13.2. Suppose that $\dim X = 1$, so that X is an elliptic curve, with zero element $x_0 \in X$. The line bundle $L = \mathscr{O}_X(x_0)$ is ample, and the homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

takes a point $x \in X$ to the line bundle $\mathscr{O}_X(x - x_0)$ corresponding to the divisor $x - x_0$; it is well-known that this is an isomorphism.

Construction of the dual abelian variety. According to the results from last time, the quotient $\hat{X} = X/K(L)$ is actually an abelian variety. So we get an isomorphism of groups $\hat{X} \cong \operatorname{Pic}^0(X)$. The abelian variety $\hat{X}$ should therefore be a "moduli space" for translation-invariant line bundles on X. What extra structure do we need to make that statement precise?

(A) We need a "universal" line bundle P on the product $X \times \hat{X}$. For every point $\alpha \in \hat{X}$, we want the line bundle
$$P_\alpha = P|_{X \times \{\alpha\}}$$
to represent the element of $\operatorname{Pic}^0(X)$ corresponding to α under the isomorphism $\hat{X} \cong \operatorname{Pic}^0(X)$. If we impose the additional condition that $P|_{\{0\}\times X}$ is trivial, then P is determined up to isomorphism (by the seesaw theorem). This line bundle is called the *Poincaré bundle*.

(B) All families of line bundles in $\operatorname{Pic}^0(X)$ should come from P, in the following sense. Suppose that S is a normal variety (for technical reasons), and that K is a line bundle on $X \times S$ such that
$$K_s = K|_{X \times \{s\}} \in \operatorname{Pic}^0(X)$$
for every $s \in S$, and such that $K|_{\{0\}\times X}$ is trivial. We then get a function
$$f\colon S \to \hat{X}$$
by sending a point $s \in S$ to the unique point $f(s) \in \hat{X}$ such that $K_s \cong P_{f(s)}$. (There is a unique point because $\hat{X} \cong \operatorname{Pic}^0(X)$ as groups.) Then we want the function f to be a morphism of varieties, and $K \cong (\operatorname{id} \times f)^* P$.

The two conditions actually determine the pair $(\hat{X}, P)$ up to isomorphism. The reason is that if we have another pair (Y, Q) with the same properties, then (B), applied to the line bundle Q on $X \times Y$, gives us a unique morphism
$$f\colon Y \to \hat{X}$$
such that $(\operatorname{id} \times f)^* P \cong Q$. For the same reason, (B) applied to the line bundle P on $X \times \hat{X}$ gives us a unique morphism
$$g\colon \hat{X} \to Y$$
such that $(\operatorname{id} \times g)^* Q \cong P$. Uniqueness then implies that $f \circ g = \operatorname{id}_{\hat{X}}$ and $g \circ f = \operatorname{id}_Y$, and so Y is isomorphic to $\hat{X}$, and the pullback of Q is isomorphic to P.

Remark. The properties above make $\hat{X}$ a so-called "fine" moduli space. This way of describing moduli spaces – where families of objects parametrized by S are in one-to-one correspondence with morphisms from S into the moduli space – is due to Grothendieck. The fact that this determines the moduli space up to isomorphism is then basically Yoneda's lemma: a scheme (or variety) is uniquely determined by knowing all morphisms from other schemes (or varieties) into it.

Now let's actually construct the dual abelian variety $\hat{X}$. As explained above, we choose an ample line bundle L on the abelian variety X, and then define
$$\hat{X} = X/K(L)$$
as the quotient by the finite subgroup $K(L) = \ker \phi_L$. Let $\pi\colon X \to \hat{X}$ be the quotient map; this is a surjective homomorphism with finite kernel, hence an isogeny. The mapping $\phi_L\colon X \to \operatorname{Pic}^0(X)$ then induces an isomorphism of groups $\hat{X} \cong \operatorname{Pic}^0(X)$.

Next, we construct the Poincaré bundle P on $X \times \hat{X}$. If we set
$$K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1},$$

then the Poincaré bundle must satisfy

$$(\mathrm{id}\times\pi)^*P \cong K.$$

This is dictated by (B), applied to the line bundle K on the product $X \times X$: we have $K_x \cong t_x^*L \otimes L^{-1} = \phi_L(x)$, and this exactly corresponds to the point $\pi(x)$ under our isomorphism $\hat{X} \cong \mathrm{Pic}^0(X)$. So the question becomes whether there is a line bundle P on $X \times \hat{X}$ such that $(\mathrm{id}\times\pi)^*P \cong K$. Now

$$\mathrm{id}\times\pi\colon X \times X \to X \times \hat{X}$$

is an isogeny with kernel $\{0\} \times K(L)$, and so according to Proposition 12.4 from last time, all we need is to lift the translation action by the finite group $\{0\} \times K(L)$ on $X \times X$ to an action on the line bundle K.

So let's take a point $a \in K(L)$ and compute:

$$\begin{aligned} t_{(0,a)}^*K &\cong t_{(0,a)}^*m^*M \otimes t_{(0,a)}^*p_1^*L^{-1} \otimes t_{(0,a)}^*p_2^*L^{-1} \\ &\cong m^*t_a^*L \otimes p_1^*L^{-1} \otimes p_2^*t_a^*L^{-1} \cong m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1} = K, \end{aligned}$$

because $t_a^*L \cong L$, due to the fact that $a \in K(L)$. This means that we can choose a collection of isomorphisms

$$\phi_a\colon t_{(0,a)}^*K \to K.$$

Each ϕ_a is of course only unique up to a nonzero constant. In order for K to be equivariant, we need $\phi_a \circ \phi_b = \phi_{a+b}$, and so we need to make the right choice of ϕ_a. This can be done as follows. Observe that

$$\begin{aligned} K|_{\{0\}\times X} &\cong m^*L|_{\{0\}\times X} \otimes p_1^*L^{-1}|_{\{0\}\times X} \otimes p_2^*L^{-1}|_{\{0\}\times X} \\ &\cong L \otimes \left(\mathscr{O}_X \otimes L^{-1}|_0\right) \otimes L \cong \mathscr{O}_X \otimes L^{-1}|_0 \end{aligned}$$

is a trivial line bundle with fiber the 1-dimensional k-vector space $L^{-1}|_0$. We can normalize each ϕ_a by requiring that it acts trivially (meaning, as the identity) on the fiber of this line bundle. This uniquely determines ϕ_a, and the uniqueness also gives $\phi_{a+b} = \phi_a \circ \phi_b$. So we get a line bundle P on $X \times \hat{X}$, unique up to isomorphism, such that

$$(\mathrm{id}\times\pi)^*P \cong m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1} \tag{13.3}$$

It is easy to see that (A) holds: write a given point $\alpha \in \hat{X}$ as $\alpha = \pi(x)$ for some $x \in X$, and observe that

$$P_\alpha = P|_{X\times\{\alpha\}} \cong (\mathrm{id}\times\pi)^*P|_{X\times\{x\}} \cong t_x^*L \otimes L^{-1},$$

which is correct because α go to $\phi_L(x)$ under our isomorphism $\hat{X} \cong \mathrm{Pic}^0(X)$.

It remains to check (B), and here we are going to use the fact that k has characteristic 0. Suppose that S is a normal variety, and that K is a line bundle on $X \times S$ with the property that

$$K_s = K|_{X\times\{s\}} \in \mathrm{Pic}^0(X)$$

and such that $K|_{\{0\}\times X}$ is trivial. We need to construct a morphism $f\colon S \to \hat{X}$ such that $K_s \cong P_{f(s)}$ for every $s \in S$. We'll do this by constructing the graph of f inside $S \times \hat{X}$. To that end, consider the line bundle

$$E = p_{12}^*K \otimes p_{13}^*(P^{-1})$$

on the product $X \times S \times \hat{X}$. For a pair $(s,\alpha) \in S \times \hat{X}$, we have

$$E|_{X\times\{s\}\times\{\alpha\}} \cong K_s \otimes P_\alpha^{-1},$$

and we want $\alpha = f(s)$ exactly when this line bundle is trivial. So let

$$\Gamma = \left\{\, (s,\alpha) \in S \times \hat{X} \;\middle|\; E \text{ is trivial on } X \times \{s\} \times \{\alpha\} \,\right\}.$$

According to Theorem 9.10, this is a closed subset of $S \times \hat{X}$. Because $K_s \in \operatorname{Pic}^0(X)$, and $\hat{X} \cong \operatorname{Pic}^0(X)$, for every $s \in S$, there is a unique point $\alpha \in \hat{X}$ such that $(s, \alpha) \in \Gamma$, and so the first projection $p_1 \colon \Gamma \to S$ is bijective. Now Γ is a reduced variety, and S is a normal variety, and because we are in characteristic zero, it follows that p_1 is birational. Because S is normal, p_1 is then an isomorphism (by Zariski's main theorem). This shows that Γ is the graph of a morphism $f \colon S \to \hat{X}$. By the seesaw theorem, the restriction of E to $X \times \Gamma$ is trivial; pulling back along the morphism $X \times S \to X \times S \times \hat{X}$, $(x, s) \mapsto (x, s, f(s))$, we then get

$$K \cong (\mathrm{id} \times f)^* P$$

as desired.