Lecture 11 (March 4)

We continue our study of line bundles on abelian varieties, based on the theorem of the cube. At the end of the previous class, we proved that if f, g, h are three morphisms from an arbitrary variety T to an abelian variety X, and if L is any line bundle on X, then

$$(11.1)\quad (f+g+h)^*L \cong (f+g)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

As a first application of this formula, we have the so-called "theorem of the square"; over the complex numbers, we already proved this back in Lecture 6.

Corollary 11.2. *Let L be a line bundle on an abelian variety, and $x, y \in X$ any two points. Then $t_{x+y}^*L \otimes L^{-1} \cong t_x^*L \otimes t_y^*L$.*

Proof. Let $f\colon \to X$ be the constant map $f \equiv x$, let $g\colon X \to X$ be the constant map $g \equiv y$, and let $h = \mathrm{id}$ be the identity. Then $f + h = t_x$, $g + h = t_y$, and $f + g + h = t_{x+y}$, and we get the desired isomorphism by applying (11.1). □

As in Lecture 6, the theorem of the square has the following interpretation. Let $\mathrm{Pic}(X)$ denote the set of isomorphism classes of line bundles on X; this is an abelian group under tensor product. Any line bundle L on X determines a function

$$\phi_L\colon X \to \mathrm{Pic}(X), \quad \phi_L(x) = t_x^*L \otimes L^{-1}.$$

The theorem of the square shows that $\phi_L(x + y) = \phi_L(x) \otimes \phi_L(y)$, and so ϕ_L is a group homomorphism. Moreover, any line bundle of the form $t_x^*L \otimes L^{-1}$ is translation-invariant, because

$$t_y^*\bigl(t_x^*L \otimes L^{-1}\bigr) = t_{x+y}^*L \otimes t_y^*L^{-1} \cong t_x^*L \otimes L^{-1}.$$

Later on, we are going to show that the set of translation-invariant line bundles is itself an abelian variety, denoted $\mathrm{Pic}^0(X)$, and that $\phi_L\colon X \to \mathrm{Pic}^0(X)$ is a morphism of abelian varieties.

Example 11.3. In terms of divisors, the theorem of the cube becomes a result about linear equivalence: for any divisor D, one has

$$t_{x+y}^*D + D \equiv t_x^*D + t_y^*D,$$

where $\equiv$ means linear equivalence. In particular, we always have

$$t_x^*D + t_{-x}^*D \equiv 2D,$$

just as in the complex case.

A second application concerns the homomorphisms

$$n_X\colon X \to X, \quad x \mapsto n \cdot x$$

and how they affect line bundles.

Corollary 11.4. *Let $n \in \mathbb{Z}$. For any line bundle L on X, one has*

$$n_X^*L \cong L^{n(n-1)/2} \otimes (-1)_X^*L^{n(n-1)/2}.$$

Proof. Take $f = (n+1)_X$, $g = 1_X$, and $h = (-1)_X$. Then $f + g + h = (n+1)_X$, $f + g = (n+2)_X$, $f + h = n_X$, and $g + h \equiv 0$, and so (11.1) gives

$$(n+1)_X^*L \cong (n+2)_X^*L \otimes n_X^*L \otimes (n+1)_X^*L^{-1} \otimes L^{-1} \otimes (-1)_X^*L^{-1}.$$

We can put this into the nicer-looking form

$$(n+2)_X^*L \otimes (n+1)_X^*L^{-2} \otimes n_X^*L \cong L \otimes (-1)_X^*L,$$

and then we recognize this as the "second difference" of the function $\mathbb{Z} \to \operatorname{Pic}(X)$, $n \mapsto n_X^* L$. Recall that if $f\colon \mathbb{Z} \to G$ is a function from the integers into an abelian group, f has degree ≤ 1 iff the first difference

$$f(n+1) - f(n) = a$$

is constant, equal to some $a \in G$; in that case, $f(n) = n \cdot a + b$, where $b = f(0)$. Similarly, f has degree ≤ 2 if the second difference

$$f(n+2) - 2f(n+1) + f(n) = a$$

is constant, and in that case, $f(n) = \binom{n}{2}a + \binom{n}{1}b + \binom{n}{0}c$ for some $b, c \in G$. Applied to our situation, this gives

$$n_X^* L \cong \bigl(L \otimes (-1)_X^* L\bigr)^{n(n-1)/2} \otimes M_1^n \otimes M_2,$$

for certain line bundles M_1, M_2, and by taking $n = 0$ and $n = 1$, one finds that $M_1 \cong L$ and $M_2 \cong \mathscr{O}_X$. Therefore

$$n_X^* L \cong \bigl(L \otimes (-1)_X^* L\bigr)^{n(n-1)/2} \otimes L^n,$$

which simplifies to the formula we wanted. □

Example 11.5. A line bundle L is called *symmetric* if $L \cong (-1)_X^* L$; this happens for example if $L = \mathscr{O}_X(D)$ for a divisor D that is invariant under the involution $x \mapsto -x$. When L is symmetric, one has

$$n_X^* L \cong L^{n^2}.$$

Similarly, L is called *anti-symmetric* if $L^{-1} \cong (-1)_X^* L$; in that case,

$$n_X^* L \cong L^n.$$

Those are the two extreme cases. Of course, for any L, the tensor product $L \otimes (-1)_X^* L$ will be symmetric, and $L^{-1} \otimes (-1)_X^* L$ will be anti-symmetric.

The homomorphism ϕ_L and ampleness. Let's look at the homomorphism

$$\phi_L\colon X \to \operatorname{Pic}(X), \quad \phi_L(x) = t_x^* L \otimes L^{-1},$$

in more detail. In the complex case, we proved that if $L = L(H, \alpha)$, with H positive definite, then ϕ_L has a finite kernel, of order $(\dim H^0(X, L))^2$. In general, ϕ_L gives us a useful way for detecting whether or not L is ample.

Definition 11.6. For a line bundle L on an abelian variety X, we define

$$K(L) = \ker \phi_L = \bigl\{\, x \in X \bigm| t_x^* L \cong L \,\bigr\}$$

which is a subgroup of the abelian group X.

In fact, $K(L)$ is also closed in the Zariski topology. To see why, consider the line bundle $m^* L \otimes p_2^* L^{-1}$ on the product $X \times X$, where $m\colon X \times X \to X$ is the group operation. For $x \in X$, we have

$$m^* L \otimes p_2^* L^{-1}|_{\{x\} \times X} \cong t_x^* L \otimes L^{-1},$$

and so the subgroup $K(L)$ can also be written in the form

$$K(L) = \bigl\{\, x \in X \bigm| m^* L \otimes p_2^* L^{-1} \text{ is trivial on } \{x\} \times X \,\bigr\}.$$

By the seesaw theorem (in Theorem 9.10), this is a closed subset of X.

Now let D be an effective divisor on X, and consider the line bundle $L = \mathscr{O}_X(D)$; in other words, we are assuming that $H^0(X, L) \neq 0$.

Theorem 11.7. *The following four conditions are equivalent:*

(a) L is ample.
(b) $K(L)$ is a finite group.

(c) The group $H = \{ x \in X \mid t_x^* D = D \}$ *is finite.*
(d) The linear system $|2D|$ *has no base points, and defines a* finite *morphism to projective space.*

Note that in (c), $t_x^* D = D$ means equality as divisors, so every irreducible component of D needs to be invariant under translation by x. The most interesting implication is that finiteness of $K(L)$ implies ampleness of L; but also note that (d) is very similar to the Lefschetz theorem (in Theorem 6.5).

Proof. Clearly $H \subseteq K(L)$, and so (b) trivially implies (c). It is also not hard to see that (d) implies (a). Indeed, the morphism $\phi_{|2D|} \colon X \to \mathbb{P}^N$ has the property that $\phi_{|2D|}^* \mathscr{O}_{\mathbb{P}^N}(1) \cong L^2$. Now the pullback of an ample line bundle by a finite morphism remains ample, and so L must be ample. (This fact is a substitute for the complex-analytic description of ampleness in terms of positive metrics.)

Let's show that (a) implies (b). We know that $K(L)$ is a closed subgroup, and so the connected component containing the point $0 \in K(L)$ is an abelian variety $Y \subseteq X$. To prove that $K(L)$ is finite, we need to show that $\dim Y = 0$. By construction, we have $t_y^* L \cong L$ for every $y \in Y$. Now consider the restriction $L_Y = L|_Y$. This is an ample line bundle on the abelian variety Y, with the property that $t_y^* L_Y \cong L_Y$ for every $y \in Y$. By the seesaw theorem (applied to the line bundle $m^* L_Y \otimes p_2^* L_Y^{-1}$ on $Y \times Y$), it follows that $m^* L_Y \otimes p_1^* L_Y^{-1} \otimes p_2^* L_Y^{-1}$ is trivial, and hence that

$$m^* L_Y \cong p_1^* L_Y \otimes p_2^* L_Y.$$

If we now pull back this identity along the mapping $Y \to Y \times Y$, $y \mapsto (y, -y)$, we get

$$\mathscr{O}_Y \cong L_Y \otimes (-1)_Y^* L_Y.$$

But both line bundles on the right-hand side are ample, and an ample line bundle on a complete variety Y can only be trivial if $\dim Y = 0$. Therefore $K(L)$ must be finite.

The most interesting implication is from (c) to (d). We already know that $|2D|$ has no base points: the reason is that

$$t_x^* D + t_{-x}^* D \equiv 2D,$$

and so for any $y \in X$, we only need to choose $x \in X$ such that $y \pm x \notin \operatorname{Supp} D$ to get a divisor linearly equivalent to $2D$ that does not pass through the point y. So we always have a morphism

$$\phi = \phi_{|2D|} \colon X \to \mathbb{P}^N,$$

where $\mathbb{P}^N$ is really the projectivization of the k-vector space $H^(X, L^2)$. We need to show that ϕ is a finite morphism. Because X is proper, ϕ is proper, and so it suffices to prove that ϕ has finite fibers. Let's argue by contradiction and assume that ϕ does not have finite fibers. Then there is an irreducible proper curve $C \subseteq X$ such that $\phi(C)$ is a point. Because the divisors in $|2D|$ correspond to hyperplanes in $\mathbb{P}^N$, and because a hyperplane either passes through a given point or is disjoint from it, we find that every divisor in $|2D|$ either contains the curve C, or is disjoint from it. In particular, for every $x \in X$, the divisor $t_x^* D + t_{-x}^* D$ either contains C, or is disjoint from C. Because C cannot be contained in all translates of D for obvious reasons, we can certainly find a point $x \in X$ such that C is disjoint from the divisor $t_x^* D$.

Now write $t_x^* D = m_1 D_1 + \cdots + m_k D_k$ as a sum of irreducible divisors. The lemma below implies that each D_j is invariant under all translations of the form $t_{x_2 - x_1}$ with $x_1, x_2 \in C$. But this clearly contradicts the finiteness of H, and so the morphism ϕ_L must have been finite after all. $\square$

Lemma 11.8. *Let E be an irreducible divisor on an abelian variety. If there is an irreducible curve C such that $E \cap C = \emptyset$, then $t_{x_1-x_2}^* E = E$ for all $x_1, x_2 \in C$.*

Proof. Consider the line bundle $L = \mathscr{O}_X(E)$. Because C is disjoint from E, the restriction $L|_C$ is trivial, and therefore has degree 0. Because the degree is constant in families, the restriction of $t_x^* L$ to C will have degree 0 for every $x \in X$. (To prove this rigorously, we can pull back to the normalization and use the Riemann-Roch theorem to express the degree in terms of the Euler characteristic; we know from Corollary 9.8 that the Euler characteristic is constant in families.) This implies that if the curve $t_x(C)$ intersects E, then it must be contained in E (because a line bundle of degree 0 with a nontrivial section is trivial).

Now let $x_1, x_2 \in C$ and $y \in E$. Then the curve $t_{y-x_2}(C)$ intersects E in the point y, and so $t_{y-x_2}(C) \subseteq E$; therefore $y + x_1 - x_2 \in E$ for every $y \in E$, which says exactly that $t_{x_1-x_2}^* E = E$. $\square$

The theorem shows that on abelian varieties, ampleness of a line bundle can be detected on curves. A very neat corollary of the theorem is that abelian varieties are always projective.

Corollary 11.9. *Every abelian variety is projective.*

Proof. Let $U \subseteq X$ be an affine open set containing the point $0 \in X$. Because X is complete and nonsingular, the complement $X \setminus U$ is a union of irreducible divisors $D_1, \dots, D_r$ (because regular functions on nonsingular varieties extend over subvarieties of codimension ≥ 2). Set $D = D_1 + \dots + D_r$. The subgroup

$$H = \left\{ x \in X \mid t_x^* D = D \right\}$$

is closed in X, and translation by any $x \in H$ preserves $U = X \setminus D$. Because $0 \in U$, this shows that $H \subseteq U$. But now H is complete and U is affine, and so H must be finite. Theorem 11.7 implies that $\mathscr{O}_X(D)$ is ample. $\square$

Torsion points. As in the complex case, we can also prove that X is always a divisible group.

Corollary 11.10. *The group X is divisible, and $X_n = \left\{ x \in X \mid n \cdot x = 0 \right\}$ is finite.*

Proof. For divisibility, we only need to prove that the homomorphism $n_X \colon X \to X$ is surjective for every $n \neq 0$. For dimension reasons, it is enough to prove that $\ker n_X$ is finite. Let L be an ample line bundle (which exists because X is projective). Then

$$n_X^* L \cong L^{n(n+1)/2} \otimes (-1)_X^* L^{n(n-1)/2},$$

and the line bundle on the right-hand side is again ample. Since an ample line bundle cannot be trivial on a complete variety of positive dimension, we find that $\ker(n_X)$ must be 0-dimensional, and therefore finite. $\square$

In the complex case, the fact that $X \cong (\mathbb{R}/\mathbb{Z})^{2g}$ made it easy to compute the kernel of n_X. We can prove somewhat similar results in general, except when the characteristic $p = \operatorname{char}(k)$ divides n.

Proposition 11.11. *Let $n \in \mathbb{Z}$ be an integer.*

(a) *The degree of n_X is equal to n^{2g}, where $g = \dim X$.*
(b) *n_X is separable iff $p \nmid n$.*
(c) *If $p \nmid n$, then $X_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.*
(d) *There is an integer $r \in \{0, 1, \dots, g\}$ such that $X_{p^e} \cong (\mathbb{Z}/p^e\mathbb{Z})^r$.*

Suppose that $f\colon X \to Y$ is a surjective morphism between two n-dimensional varieties. The extension of function fields

$$k(Y) \subseteq k(X)$$

is finite algebraic, and we define $\deg f = \big(k(X)\colon k(Y)\big)$. When f is separable, meaning when the field extension is separable, the number of points in the fiber $f^{-1}(y)$ is equal to $\deg f$ for most $y \in Y$. (More precisely, there is a nonempty Zariski-open subset of Y where this is true.) When f is inseparable, we define the *separable degree* of f as the separable degree of the field extension $k(Y) \subseteq k(X)$; then the number of points in the general fiber is equal to the separable degree.

Recall from the complex case that an *isogeny* $f\colon X \to Y$ is a surjective homomorphism between two abelian varieties whose kernel is finite. The typical examples are the homomorphisms $n_X\colon X \to X$ with $n \neq 0$. In the case of an isogeny, all fibers have the same number of points; therefore the number of points in $X_n = \ker(n_X)$ is equal to the separable degree of n_X. We'll compute this degree next time.