## Lecture 10: March 6

**Algebraic $\mathscr{D}$-modules.** Let me first recall the definition of an algebraic $\mathscr{D}$-module from last time. As before, $X$ is an algebraic variety over a field $k$, nonsingular of constant dimension $n$. We denote by $\mathscr{D}_X$ the sheaf of algebraic differential operators on $X$, and by $F_j\mathscr{D}_X$ the subsheaf of operators of order $\leq j$. Then each $F_j\mathscr{D}_X$ is a coherent sheaf of $\mathscr{O}_X$-modules, and $\mathscr{D}_X$ itself is quasi-coherent.

**Definition 10.1.** An *algebraic $\mathscr{D}$-module* is a quasi-coherent sheaf of $\mathscr{O}_X$-modules $\mathcal{M}$, together with a left (or right) action by $\mathscr{D}_X$.

Since $\mathscr{D}_X$ is noncommutative, we again have to distinguish between left and right modules. In the case of a left $\mathscr{D}$-module $\mathcal{M}$, the set of sections $M = \Gamma(U, \mathcal{M})$ over any affine open subset $U \subseteq X$ is thus a left module over the algebra of differential operators $D(A)$, where $A = \Gamma(U, \mathscr{O}_X)$. The quasi-coherence condition means that the restriction of $\mathcal{M}$ to the open set $U$ is uniquely determined by this $D(A)$-module. Recall from Lecture 9 that the algebra $D(A)$ is generated, as an $A$-subalgebra of $\operatorname{End}_k(A)$, by the derivations $\operatorname{Der}_k(A)$, subject to the relation $[\delta, f] = \delta(f)$ for all $\delta \in \operatorname{Der}_k(A)$ and all $f \in A$. The left $D(A)$-action on $M$ is therefore the same thing as a $k$-linear mapping

$$\operatorname{Der}_k(A) \otimes_k M \to M, \quad \delta \otimes m \mapsto \delta m,$$

such that $(f\delta)m = f(\delta m)$, $\delta(fm) = f\delta(m) + \delta(f)m$ and $\delta(\eta m) - \eta(\delta m) = [\delta, \eta]m$ for all $\delta, \eta \in \operatorname{Der}_k(A)$, all $f \in A$, and all $m \in M$. Globally, to turn a quasi-coherent sheaf of $\mathscr{O}_X$-modules $\mathcal{M}$ into a left $\mathscr{D}_X$-module, we need a $k$-linear morphism

$$\mathscr{T}_X \otimes_k \mathcal{M} \to \mathcal{M}$$

that satisfies those three conditions locally. (You can work out for yourself what happens for right $\mathscr{D}$-modules.)

*Example* 10.2. Since the algebra of differential operators on the affine space $\mathbb{A}_k^n$ is the Weyl algebra $A_n(k)$, an algebraic $\mathscr{D}$-module on $\mathbb{A}_k^n$ is (up to the equivalence between quasi-coherent sheaves and modules) the same thing as a left (or right) module over $A_n(k)$.

Here are some examples of left and right $\mathscr{D}$-modules.

*Example* 10.3. The structure sheaf $\mathscr{O}_X$ is a left $\mathscr{D}_X$-module. Indeed, for every affine open subset $U \subseteq X$, the algebra of differential operators $D(A)$ acts on $A = \Gamma(U, \mathscr{O}_X)$ by construction.

*Example* 10.4. Every algebraic vector bundle with integrable connection is a left $\mathscr{D}_X$-module. Let $\mathscr{E}$ be the corresponding locally free sheaf of $\mathscr{O}_X$-modules; in Hartshorne's notation, the vector bundle is then $\mathbb{V}(\mathscr{E}^*)$. A *connection* is a $k$-linear morphism $\nabla \colon \mathscr{E} \to \Omega^1_{X/k} \otimes_{\mathscr{O}_X} \mathscr{E}$ that satisfies the Leibniz rule. In other words, for every affine open subset $U \subseteq X$ and every pair of sections $s \in \Gamma(U, \mathscr{E})$ and $f \in \Gamma(U, \mathscr{O}_X)$, the connection should satisfy

$$\nabla(fs) = f\nabla(s) + df \otimes s.$$

We can also regard the connection as a $k$-linear morphism $\nabla \colon \mathscr{T}_X \otimes_k \mathscr{E} \to \mathscr{E}$, but we use the differential geometry notation $\nabla_\theta(s)$ instead of $\nabla(\theta \otimes s)$ for $\theta \in \Gamma(U, \mathscr{T}_X)$ and $s \in \Gamma(U, \mathscr{E})$. In this notation, we have

$$\nabla_{f\theta}(s) = f\nabla_\theta(s), \tag{10.5}$$

and the Leibniz rule becomes

$$\nabla_\theta(fs) = f\nabla_\theta(s) + \theta(f)s. \tag{10.6}$$

The connection is called *integrable* if

$$\nabla_\theta \circ \nabla_\eta - \nabla_\eta \circ \nabla_\theta = \nabla_{[\theta,\eta]} \tag{10.7}$$

for every pair of vector fields $\theta, \eta \in \Gamma(U, \mathscr{T}_X)$. This is equivalent to the vanishing of the curvature operator in $\Omega^2_{X/k} \otimes_{\mathscr{O}_X} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$. The conditions in (10.5), (10.6) and (10.7) are exactly saying that the action of $\mathscr{T}_X$ on $\mathscr{E}$ extends to a left action by the sheaf of differential operators $\mathscr{D}_X$, and so $\mathscr{E}$ becomes a left $\mathscr{D}$-module.

In general, the left action of $\mathscr{D}_X$ on a left $\mathscr{D}$-module $\mathcal{M}$ may be considered (formally) as a connection operator $\nabla\colon \mathcal{M} \to \Omega^1_{X/k} \otimes_{\mathscr{O}_X} \mathcal{M}$ that satisfies the Leibniz rule and is integrable, in the sense that it locally satisfies the conditions expressed in (10.5), (10.6) and (10.7).

*Example* 10.8. Unlike in the case of affine space, we cannot turn left $\mathscr{D}$-modules into right $\mathscr{D}$-modules by changing signs, since we might not be able to do this consistently on all affine open subsets. Instead, the primary example of a right $\mathscr{D}$-module is the canonical bundle $\omega_X = \bigwedge^n \Omega^1_{X/k}$, whose sections are the algebraic $n$-forms. If $U \subseteq X$ is an affine open subset with local coordinates $x_1, \dots, x_n$, then $\omega_X$ is locally free of rank one, spanned by $dx_1 \wedge \dots \wedge dx_n$. The tangent sheaf $\mathscr{T}_X$ acts on $\omega_X$ by Lie differentiation. Given $\omega \in \Gamma(U, \omega_X)$ and $\theta, \theta_1, \dots, \theta_n \in \Gamma(U, \mathscr{T}_X)$, the formula for the Lie derivative is

$$(\operatorname{Lie}_\theta \omega)(\theta_1, \dots, \theta_n) = \theta \cdot \omega(\theta_1, \dots, \theta_n) - \sum_{i=1}^n \omega\bigl(\theta_1, \dots, [\theta, \theta_i], \dots, \theta_n\bigr).$$

One can check quite easily that the following relations hold:

$$\operatorname{Lie}_\theta(f\omega) = f \operatorname{Lie}_\theta \omega + \theta(f)\omega = \operatorname{Lie}_{f\theta} \omega$$
$$\operatorname{Lie}_{[\theta,\eta]} \omega = \operatorname{Lie}_\theta \operatorname{Lie}_\eta \omega - \operatorname{Lie}_\eta \operatorname{Lie}_\theta \omega$$

This almost looks like $\omega_X$ should be a left $\mathscr{D}_X$-module, but note that (10.5) is not satisfied since $\operatorname{Lie}_{f\theta} \omega \neq f \operatorname{Lie}_\theta \omega$. But if we instead define

$$\omega_X \otimes_k \mathscr{T}_X \to \omega_X, \quad \omega \otimes \theta \mapsto \omega \cdot \theta = -\operatorname{Lie}_\theta(\omega)$$

and also write the $\mathscr{O}_X$-action on $\omega_X$ on the right, we obtain

$$\omega \cdot \theta(f) = (-\operatorname{Lie}_\theta \omega) f + \operatorname{Lie}_\theta(\omega f) = (\omega \cdot \theta) f - (\omega f) \cdot \theta$$
$$\omega \cdot [\theta, \eta] = -\operatorname{Lie}_{[\theta,\eta]} \omega = \operatorname{Lie}_\theta \operatorname{Lie}_\eta \omega - \operatorname{Lie}_\eta \operatorname{Lie}_\theta \omega = (\omega \cdot \theta) \cdot \eta - (\omega \cdot \eta) \cdot \theta.$$

These are exactly the relations defining $\mathscr{D}_X$, and so we obtain on $\omega_X$ the structure of a right $\mathscr{D}_X$-module. In local coordinates, we have

$$(f\, dx_1 \wedge \dots \wedge dx_n) \cdot P = (P^\sigma f) dx_1 \wedge \dots \wedge dx_n,$$

where $P^\sigma = \sum (-\partial)^\alpha f_\alpha$ is the formal adjoint of $P = \sum f_\alpha \partial^\alpha$. In local coordinates, the left $\mathscr{D}$-module structure on $\mathscr{O}_X$ and the right $\mathscr{D}$-module structure on $\omega_X$ are therefore related to each other exactly as in the case of the Weyl algebra.

**Good filtrations and characteristic variety.** As in the case of the Weyl algebra, we study $\mathscr{D}$-modules using filtrations. Let $\mathcal{M}$ be a left $\mathscr{D}_X$-module. We consider increasing filtrations $F_\bullet \mathcal{M}$ by coherent $\mathscr{O}_X$-submodules $F_j \mathcal{M}$ such that

$$F_i \mathscr{D}_X \cdot F_j \mathcal{M} \subseteq F_{i+j} \mathcal{M}$$

for all $i, j \in \mathbb{Z}$. We also assume that the filtration is exhaustive, meaning that

$$\bigcup_{j \in \mathbb{Z}} F_j \mathcal{M} = \mathcal{M}.$$

Note that each $F_j\mathcal{M}$ is assumed to be coherent over $\mathscr{O}_X$. We say that such a filtration is *good* if the associated graded module

$$\operatorname{gr}^F\mathcal{M} = \bigoplus_{j\in\mathbb{Z}} F_j\mathcal{M}/F_{j-1}\mathcal{M}$$

is locally finitely generated over $\operatorname{gr}^F\mathscr{D}_X$. This implies that $F_j\mathcal{M} = 0$ for $j \ll 0$.

Now suppose that $U \subseteq X$ is an affine open subset, and set $A = \Gamma(U, \mathscr{O}_X)$ and $M = \Gamma(U, \mathcal{M})$. By the same argument as in the case of the Weyl algebra, one shows that $M$ is finitely generated over $D(A)$ if and only if admits a good filtration $F_\bullet M$ by finitely generated $A$-modules; again, this means that $F_iD(A) \cdot F_jM \cdot F_{i+j}M$ and $\operatorname{gr}^F M$ is finitely generated over $\operatorname{gr}^F D(A)$.

**Definition 10.9.** We say that a left (or right) $\mathscr{D}_X$-module is *coherent* if it is locally finitely generated over $\mathscr{D}_X$.

Note that this is not the same thing as being $\mathscr{O}_X$-coherent; in fact, most coherent $\mathscr{D}_X$-modules are not coherent over $\mathscr{O}_X$. Every coherent $\mathscr{D}_X$-module has a good filtration locally, meaning on each affine open subset; in fact, we will see next time that coherent $\mathscr{D}_X$-modules always admit a global good filtration $F_\bullet\mathcal{M}$.

Given a good filtration $F_\bullet\mathcal{M}$ (globally or locally), the associated graded $\operatorname{gr}^F\mathcal{M}$ is coherent over the sheaf of $\mathscr{O}_X$-algebras

$$\operatorname{gr}^F\mathscr{D}_X \cong \operatorname{Sym}\mathscr{T}_X \cong p_*\mathscr{O}_{T^*X},$$

where $p\colon T^*X \to X$ again means the cotangent bundle. By the correspondence between coherent sheaves on $T^*X$ and finitely generated modules over $p_*\mathscr{O}_{T^*X}$, we thus obtain a coherent sheaf of $\mathscr{O}_{T^*X}$-modules on the cotangent bundle that we denote by the symbol $\widetilde{\operatorname{gr}^F\mathcal{M}}$.

**Definition 10.10.** The *characteristic variety* $\operatorname{Ch}(\mathcal{M})$ is the closed algebraic subset of $T^*X$ given by the support of $\widetilde{\operatorname{gr}^F\mathcal{M}}$, with the reduced scheme structure.

As in the case of the Weyl algebra, any two good filtrations on $\mathcal{M}$ are comparable; for the same reason as before, this implies that the subsheaf

$$\sqrt{\operatorname{Ann}_{\operatorname{gr}^F\mathscr{D}_X}\operatorname{gr}^F\mathcal{M}} \subseteq \operatorname{gr}^F\mathscr{D}_X$$

is independent of the choice of good filtration. If we denote by $\mathcal{J}_\mathcal{M} \subseteq \mathscr{O}_{T^*X}$ the corresponding coherent sheaf of ideals on the cotangent bundle, then $\operatorname{Ch}(\mathcal{M})$ is the closed subscheme defined by $\mathcal{J}_\mathcal{M}$. We are going to show later on that Bernstein's inequality carries over to arbitrary coherent $\mathscr{D}$-modules: as long as $\mathcal{M} \neq 0$, every irreducible component of $\operatorname{Ch}(\mathcal{M})$ has dimension at least $n$.

*Example* 10.11. If $\mathscr{E}$ is the left $\mathscr{D}_X$-module determined by a vector bundle with integrable connection, then $\operatorname{Ch}(\mathscr{E})$ is the zero section. The reason is that $\mathscr{E}$ is coherent over $\mathscr{O}_X$, which means that setting $F_j\mathscr{E} = 0$ for $j < 0$ and $F_j\mathscr{E} = \mathscr{E}$ for $j \geq 0$ gives a good filtration. Here

$$\operatorname{Ann}_{\operatorname{gr}^F\mathscr{D}_X}\operatorname{gr}^F\mathscr{E} = \bigoplus_{j\geq 1}\operatorname{gr}_j^F\mathscr{D}_X,$$

and so $\mathcal{J}_\mathscr{E}$ is the ideal of the zero section. Of course, this works more generally for any $\mathscr{D}$-module that is coherent over $\mathscr{O}_X$.

The example has a useful converse.

**Proposition 10.12.** *Let $\mathcal{M}$ be a coherent $\mathscr{D}_X$-module. If $\mathcal{M}$ is coherent over $\mathscr{O}_X$, then $\mathcal{M}$ is actually a locally free $\mathscr{O}_X$-module of finite rank (and therefore comes from a vector bundle with integrable connection).*

*Proof.* Since $\mathcal{M}$ is a quasi-coherent $\mathscr{O}_X$-module, it suffices to check that the localization $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_X} \mathcal{M}$ at every closed point $x \in X$ is a free $\mathscr{O}_{X,x}$-module of finite rank. This reduces the problem to the following special case: $A$ is a regular local ring of dimension $n$, containing a field $k$, with maximal ideal $\mathfrak{m}$ and residue field $A/\mathfrak{m} \cong k$, and $M$ is a left $D(A)$-module that is finitely generated over $A$. Here $D(A)$ is again the algebra of $k$-linear differential operators on $A$. We need to prove that $M$ is a free $A$-module of finite rank.

First, some preparations. Since $A$ is regular of dimension $n$, the maximal ideal $\mathfrak{m}$ is generated by $n$ elements $x_1, \dots, x_n$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $k$. Let $\partial_1, \dots, \partial_n \in \operatorname{Der}_k(A)$ be the corresponding derivations, which freely generate $\operatorname{Der}_k(A)$ as an $A$-module. For every nonzero $f \in A$, we define the order of vanishing as

$$\operatorname{ord}(f) = \max\{ \ell \geq 0 \mid f \in \mathfrak{m}^\ell \};$$

this makes sense because the intersection of all powers of the maximal ideal is trivial. If $f = 0$, we formally set $\operatorname{ord}(f) = +\infty$. The key point is that we can reduce the order of vanishing of $f$ by applying a suitable derivation. Indeed, suppose that $\operatorname{ord}(f) = \ell$. The ideal $\mathfrak{m}^\ell$ is generated by all monomials of degree $\ell$ in $x_1, \dots, x_n$, and so we can write

$$f = \sum_{|\alpha|=\ell} f_\alpha x^\alpha,$$

with at least one $f_\alpha \in A$ being a unit (because otherwise $f \in \mathfrak{m}^{\ell+1}$). Choose a multi-index $\alpha$ such that $f_\alpha$ is a unit, and then choose $i = 1, \dots, n$ such that $\alpha_i \geq 1$. Since $\partial_i(x_j) = \delta_{i,j}$, we get

$$\partial_i(f) = \sum_{|\alpha|=\ell} \Big( \partial_i(f_\alpha) x^\alpha + f_\alpha \alpha_i x^{\alpha - e_i} \Big),$$

and this expression clearly belongs to $\mathfrak{m}^{\ell-1}$ but not to $\mathfrak{m}^\ell$. Hence $\operatorname{ord}(\partial_i(f)) = \ell - 1$.

As I said, we need to prove that $M$ is a free $A$-module of finite rank. To do this, pick a minimal set of generators $m_1, \dots, m_r \in M$, whose images in $M/\mathfrak{m}M$ are linearly independent over $k$. This gives us a surjective morphism of $A$-modules

$$A^{\oplus r} \to M, \quad (f_1, \dots, f_r) \mapsto f_1 m_1 + \dots + f_r m_r,$$

and we are going to show that it is also injective, hence an isomorphism. Suppose that there was a nontrivial relation $f_1 m_1 + \dots + f_r m_r = 0$. Then $f_1, \dots, f_r \in \mathfrak{m}$, because $m_1, \dots, m_r$ are linearly independent modulo $\mathfrak{m}M$. In other words, we have

$$\ell = \min\bigl\{ \operatorname{ord}(f_1), \dots, \operatorname{ord}(f_r) \bigr\} \geq 1.$$

Now the idea is to use the $D(A)$-module structure to create another relation for which the value of $\ell$ is strictly smaller. By repeating this, we eventually arrive at a relation with $\ell = 0$, contradicting the fact that $m_1, \dots, m_r$ are linearly independent modulo $\mathfrak{m}M$. Here we go. If we apply $\partial_i$ to our relation, we obtain

$$0 = \partial_i \cdot \sum_{j=1}^r f_j m_j = \sum_{j=1}^r [\partial_i, f_j] m_j + \sum_{j=1}^r f_j (\partial_i m_j) = \sum_{j=1}^r \partial_i(f_j) m_j + \sum_{j=1}^r f_j (\partial_i m_j).$$

We can write each $\partial_i m_j$ in terms of the generators $m_1, \dots, m_r$ as

$$\partial_i m_j = \sum_{k=1}^r a_{i,j,k} m_k,$$

and after reindexing, we get the new relation

$$\sum_{j=1}^r \Big( \partial_i(f_j) + \sum_{k=1}^r a_{i,k,j} f_k \Big) m_j = 0.$$

If we now choose $j$ such that $\operatorname{ord}(f_j) = \ell$, and then choose $i$ such that $\operatorname{ord}(\partial_i(f_j)) = \ell - 1$, then the $j$-th coefficient in the new relation belongs to $\mathfrak{m}^{\ell-1}$ but not to $\mathfrak{m}^{\ell}$, as desired. $\square$

We showed in Lecture 5 that $\mathcal{M}$ is coherent over $\mathscr{O}_X$ if and only if its characteristic variety is contained in the zero section of the cotangent bundle. This means that if $\mathcal{M}$ is a coherent $\mathscr{D}_X$-module with $\operatorname{Ch}(\mathcal{M})$ contained in the zero section, then $\mathcal{M}$ is a locally free $\mathscr{O}_X$-module of finite rank, and the $\mathscr{D}_X$-module structure is the same as the datum of an integrable connection on $\mathcal{M}$.