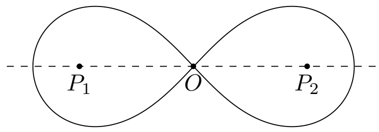
Lecture 1 (January 28)

Introduction. Our topic this semester is abelian varieties. As you probably know, abelian varieties are the higher-dimensional generalization of elliptic curves: smooth projective varieties that have the structure of an abelian group, with the group operations given by algebraic morphisms. During the first half of the semester, we will cover the basic theory, both from the analytic (= complex manifolds) and algebraic (= projective algebraic varieties) point of view. Our main source will be Mumford's book *Abelian Varieties*. After that, I plan to talk about derived categories and the Fourier transform, and about Deligne's theorem on absolute Hodge classes. I will try to provide notes for each lecture.

The lemniscate. Let's start with a bit of historical material, in order to understand where elliptic curves come from. (If you are interested in learning more about this, I recommend the article "The arithmetic-geometric mean of Gauss" by David Cox.) The length of a circular arc is easily computed with the help of trigonometric functions (and their inverses). But trying to compute the arc length of other curves such as ellipses leads to more complicated integrals, and the study of these integrals eventually led to the birth of elliptic curves. One particularly nice example is the *lemniscate*. It is defined as the set of points for which the product of the distances to two given points P_1 and P_2 (called the "foci") is constant.



In polar coordinates (r, θ), the equation of the lemniscate is $r^2 = a^2 \cos(2\theta)$, where $2a$ is the diameter of the lemniscate. The arc length of the lemniscate was first computed by the Bernoullis at the end of the 17-th century. We can easily derive their formula. If we write

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

and use r as the parameter, then $r \in [0, a]$ gives us exactly one quarter of the lemniscate. Therefore the length of the entire lemniscate is

$$L(a) = 4 \int_0^a \sqrt{\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2} \, dr.$$

Taking derivatives of our parametrization, we get

$$\frac{dx}{dr} = \cos \theta - r \sin \theta \frac{d\theta}{dr} \quad \text{and} \quad \frac{dy}{dr} = \sin \theta + r \cos \theta \frac{d\theta}{dr},$$

and so the expression inside the square root is

$$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 = 1 + r^2 \left(\frac{d\theta}{dr}\right)^2.$$

From the equation $r^2 = a^2 \cos(2\theta)$, we obtain

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{r^2}{a^4 \sin^2(2\theta)} = \frac{r^2}{a^4(1 - r^4/a^4)} = \frac{r^2}{a^4 - r^4},$$

and after substituting this into the integral and simplifying, we arrive at

$$L(a) = 4 \int_0^a \sqrt{\frac{a^4}{a^4 - r^4}} \, dr = 4a \int_0^1 \frac{dt}{\sqrt{1 - t^4}}.$$

Note that the integral looks a bit similar to

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \arcsin(1) = \frac{\pi}{2}.$$

Probably for that reason, Gauss introduced the notation

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{\varpi}{2},$$

because the symbol ϖ (LaTeX code `\varpi`) is a cursive variant of the letter pi. Gauss came across this integral in his study of the arithmetic-geometric mean. For two positive real numbers $a, b > 0$, the arithmetic-geometric mean $M(a, b)$ is the common limit of the two sequences a_n, b_n, defined recursively by

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

The two sequences converge very rapidly, and Gauss arrived at the identity

$$M\big(\sqrt{2}, 1\big) = \frac{\pi}{\varpi}$$

by computing both sides to 11 digits (by hand). This identity can be used to compute ϖ efficiently.

Let's now consider the arc length of the lemniscate

$$\int_0^x \frac{dt}{\sqrt{1-t^4}}$$

as a function of $x \in [-1, 1]$. It is obviously increasing, and takes values in the interval $[-\varpi/2, \varpi/2]$. The inverse function

$$\mathrm{sl}\colon [-\varpi/2, \varpi/2] \to [-1, 1]$$

is called the *lemniscate sine*; its defining property is that

$$\int_0^{\mathrm{sl}\, x} \frac{dt}{\sqrt{1-t^4}} = x.$$

The reason for the name is the obvious analogy with the arc sine function

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

and its inverse. We have $\mathrm{sl}(0) = 0$, $\mathrm{sl}(\varpi/2) = 1$, and $\mathrm{sl}(-\varpi/2) = -1$. Just like the sine function, the lemniscate also satisfies an addition formula. The precise result is due to Euler, I believe, but other people had already found similar addition formulas for the arc lengths of other curves (such as $y = x^3$ or ellipses).

Proposition 1.1. *Suppose that x, y, z are related by the fact that*

$$\int_0^x \frac{dt}{\sqrt{1-t^4}} + \int_0^y \frac{dt}{\sqrt{1-t^4}} = \int_0^z \frac{dt}{\sqrt{1-t^4}}.$$

Then x, y, z also satisfy the following algebraic equation:

$$z = \frac{y\sqrt{1-x^4} + x\sqrt{1-y^4}}{1 + x^2 y^2}$$

The interesting point is that the arc length function is *transcendental* (just like the trigonometric functions), but the three values in the formula above are nevertheless related by an *algebraic* equation.

Proof. This is very similar to the proof of the addition formula for sine. Let's think of $z = z(x, y)$ as a function of the two variables x and y; this makes sense when x and y are not too large, because the arc length function has an inverse. Each level set of z is a curve, and by differentiating the relation between x, y, z, we see that

$$\frac{dx}{\sqrt{1-x^4}} + \frac{dy}{\sqrt{1-y^4}} = 0$$

along this curve. Choose a local parametrization $x = x(t)$ and $y = y(t)$ for the curve, in such a way that

$$\frac{dx}{dt} = \sqrt{1-x^4} \quad \text{and} \quad \frac{dy}{dt} = -\sqrt{1-y^4}.$$

For simplicity, let's use a dot to denote the derivative with respect to t. Then

$$\dot{x}^2 = \left(\frac{dx}{dt}\right)^2 = 1 - x^4,$$

and therefore $2\dot{x}\ddot{x} = -4x^3\dot{x}$ or $\ddot{x} = -2x^3$. For the same reason, $\ddot{y} = -2y^3$. Now the trick, which is hard to guess unless you know the proof of the addition formula for sine, is to compute

$$\frac{d}{dt}(y\dot{x} - x\dot{y}) = y\ddot{x} - x\ddot{y} = 2xy(y^2 - x^2).$$

From the formulas for the first derivatives, we also have

$$(y\dot{x} - x\dot{y})(y\dot{x} + x\dot{y}) = y^2\dot{x}^2 - x^2\dot{y}^2 = (y^2 - x^2)(1 + x^2y^2).$$

After dividing both lines, we obtain

$$\frac{d}{dt}\log(y\dot{x} - x\dot{y}) = \frac{\frac{d}{dt}(y\dot{x} - x\dot{y})}{y\dot{x} - x\dot{y}} = \frac{2xy(y\dot{x} + x\dot{y})}{1 + x^2y^2} = \frac{d}{dt}\log(1 + x^2y^2).$$

After integration, this becomes

$$C(1 + x^2y^2) = y\dot{x} - x\dot{y} = y\sqrt{1-x^4} + x\sqrt{1-y^4}$$

for some constant C, and by setting $y = 0$ and $x = z$, we find that $C = z$. This gives the desired algebraic relation between x, y, z. $\square$

If we rewrite the addition formula in terms of $\operatorname{sl} x$, it becomes

$$\operatorname{sl}(x+y) = \frac{\operatorname{sl} y\sqrt{1 - \operatorname{sl}^4 x} + \operatorname{sl} x\sqrt{1 - \operatorname{sl}^4 y}}{1 + \operatorname{sl}^2 x \operatorname{sl}^2 y}. \tag{1.2}$$

Remembering that $\operatorname{sl}(\varpi/2) = 1$, we deduce that

$$\operatorname{sl}(x + \varpi/2) = \frac{\sqrt{1 - \operatorname{sl}^4 x}}{1 + \operatorname{sl}^2 x} = \sqrt{\frac{1 - \operatorname{sl}^2 x}{1 + \operatorname{sl}^2 x}},$$

at least for those values of x where both sides are defined. We can now try to extend the domain of definition. To eliminate the (potentially ambiguous) square root, we rewrite the formula above as

$$\operatorname{sl}^2(x + \varpi/2) = \frac{1 - \operatorname{sl}^2 x}{1 + \operatorname{sl}^2 x}. \tag{1.3}$$

Applying the formula to itself, we get $\operatorname{sl}^2(x+\varpi) = \operatorname{sl}^2(x)$, and therefore $\operatorname{sl}(x+\varpi) = \pm \operatorname{sl} x$. As $\operatorname{sl}(\varpi/2) = 1$ and $\operatorname{sl}(-\varpi/2) = -1$, we have to choose the minus sign if we want things to be consistent, and so

$$\operatorname{sl}(x + \varpi) = -\operatorname{sl} x.$$

This allows us to extend the lemniscate sine to a function $\operatorname{sl}\colon \mathbb{R} \to [-1, 1]$ that is periodic with period 2ϖ.

Gauss was the first person to consider the lemniscate sine as a function of a *complex* variable; this was taking place around 1800. The integral

$$\int_0^z \frac{dt}{\sqrt{1-t^4}}$$

makes sense for complex numbers $z \in \mathbb{C}$ with $|z| < 1$, by using the standard branch of the square root function. It is again invertible, at least in a neighborhood of the origin, and we denote the inverse function by the same symbol sl. The substitution $t \mapsto it$ proves the formula

$$\int_0^{iz} \frac{dt}{\sqrt{1-t^4}} = i \int_0^z \frac{dt}{\sqrt{1-t^4}},$$

and so we have $\operatorname{sl}(iz) = i \operatorname{sl}(z)$. The addition formula in (1.2) shows that

$$\operatorname{sl}(x+iy) = \frac{\operatorname{sl} x \sqrt{1-\operatorname{sl}^4 y} + i \operatorname{sl} y \sqrt{1-\operatorname{sl}^4 x}}{1-\operatorname{sl}^2 x \operatorname{sl}^2 y}$$

for $x, y \in [-\varpi/2, \varpi/2]$, and so our complex-valued lemniscate sine is defined on the square of side length ϖ centered at the origin. From this formula, we can see that $\operatorname{sl} z$ has a pole when $\operatorname{sl} x = \pm 1$ and $\operatorname{sl} y = \pm 1$, hence at the points $(\pm 1 \pm i)/2$. It also has a unique zero at the point $z = 0$. For the same reason as before, the function $\operatorname{sl} z$ satisfies the two identities

$$\operatorname{sl}(z+\varpi) = -\operatorname{sl}(z) \quad \text{and} \quad \operatorname{sl}(z+i\varpi) = -\operatorname{sl}(z),$$

and this allows us to extend $\operatorname{sl} z$ to a well-defined meromorphic function on the entire complex plane. It has simple zeros at the point $\mathbb{Z}\varpi + \mathbb{Z}i\varpi$ and simple poles at the points $\varpi(1+i)/2 + \mathbb{Z}\varpi + \mathbb{Z}i\varpi$. Note that $\operatorname{sl} z$ has two independent periods, namely 2ϖ and $(1+i)\varpi$. Unlike the usual trigonometric function, the lemniscate sine is doubly periodic. Such functions are also called *elliptic functions*. Gauss did not publish any of his results, and it took almost 30 years until Abel and Jacobi developed the theory of elliptic functions.

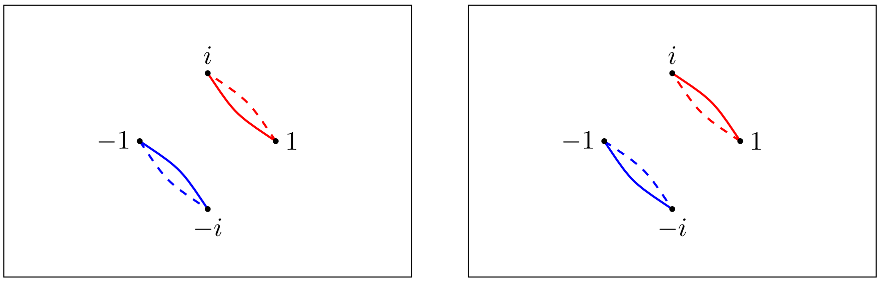
Geometric interpretation. We can interpret the above results about the integral

$$\int \frac{dt}{\sqrt{1-t^4}}$$

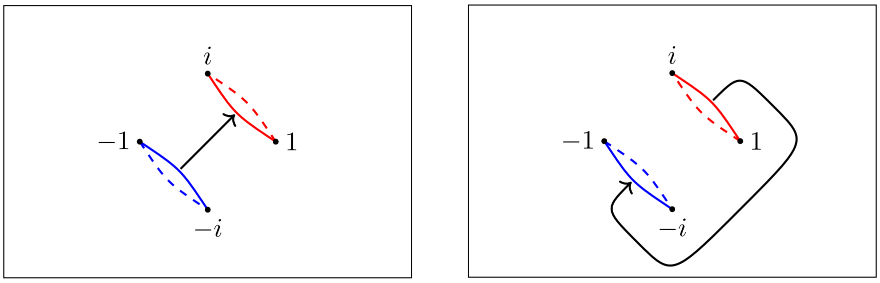
in terms of compact Riemann surfaces. Since the square-root function $\sqrt{1-x^4}$ has two branches, we introduce a new variable y such that $y^2 = 1 - x^4$, and then consider the one-form dx/y instead of $dt/\sqrt{1-t^4}$. This rule actually defines a double covering of $\mathbb{P}^1$. To see why, let's use the coordinate $u = 1/x$ on the chart at infinity, and define $v^2 = 1 - u^4$. If we glue the two according to the rule $v = -u^2 y$, then we get a compact Riemann surface C together with a two-to-one map $C \to \mathbb{P}^1$ that is branched at the four points ± 1 and $\pm i$. Moreover, dx/y is a well-defined one-form on C because

$$\frac{dx}{y} = \frac{-du/u^2}{-v/u^2} = \frac{du}{v}.$$

We can visualize the double covering as follows. Take two copies of the complex plane (or the Riemann sphere), make branch cuts from the point 1 to the point i and from the point -1 to the point $-i$, and then glue the two copies together as indicated in the picture below. This produces a surface with one handle, which means that the Riemann surface C has genus 1, hence is a torus.



We can integrate dx/y along paths in C to obtain a multi-valued holomorphic function on C. It is multi-valued because C is not simply connected. The ambiguity in the values is given by the integrals of dx/y along the two basic closed loops in C. The first loop goes around the torus.



By moving the contour of integration in both copies of the plane, one can see that the integral of dx/y over this loop is equal to

$$\int_{-1}^{1} \frac{dt}{\sqrt{1-t^4}} - \int_{1}^{-1} \frac{dt}{\sqrt{1-t^4}} = 2\varpi.$$

Similarly, the loop that goes around the neck of the torus leads to the integral $(1+i)\varpi$. The inverse of this multi-valued function will therefore be doubly periodic, with periods 2ϖ and $(1+i)\varpi$.

Elliptic curves. Let's now consider elliptic functions in general. An elliptic function is holomorphic (actually, meromorphic) function on $\mathbb{C}$ with two linearly independent periods. If we call the two periods γ_1 and γ_2, then the subgroup $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ is a discrete subgroup of $\mathbb{C}$ of rank 2. The quotient $\mathbb{C}/\Gamma$ is compact, and so Γ is called a lattice. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a holomorphic function with periods in Γ, meaning that $f(z+\gamma) = f(z)$ for every $\gamma \in \Gamma$. Because $\mathbb{C}/\Gamma$ is compact, f must be bounded, hence constant. In order to get interesting elliptic functions, we have to allow poles.

The most basic elliptic function is the *Weierstrass $\wp$-function.* The symbol $\wp$ (LaTeX code `\wp`) is the old handwritten German p, but for simplicity, I will use the regular letter P instead. Consider the infinite series

$$P(z) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right).$$

It converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \Gamma$; the reason is that, for z in a compact set, each term is bounded by a constant times $1/|\gamma|^3$, and it is easy to see that the series

$$\sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{|\gamma|^3}$$

is convergent. Therefore $P(z)$ is a well-defined meromorphic function with a double pole at each point of Γ; it is clearly even because $P(-z) = P(z)$. To show that it is Γ-periodic, we consider the derivative

$$P'(z) = \sum_{\gamma \in \Gamma} \frac{-2}{(z-\gamma)^3}.$$

The series again converges absolutely and uniformly on compact subsets, and $P'(z)$ is visibly Γ-periodic. This means that

$$P(z+\gamma) = P(z) + C(\gamma)$$

for some constant $C(\gamma) \in \mathbb{C}$. By putting $z = -\gamma/2$ and remembering that $P(z)$ is even, we get $C(\gamma) = 0$, and so $P(z)$ is itself Γ-peridic.

Let's compute the Laurent series around $z = 0$. Here we use the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{and} \quad \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

From the definition of $P(z)$, we get

$$\begin{aligned} P(z) &= \frac{1}{z^2} + \sum_{\gamma \neq 0} \frac{1}{\gamma^2} \left(\frac{1}{(1-z/\gamma)^2} - 1 \right) = \frac{1}{z^2} + \sum_{\gamma \neq 0} \sum_{n=1}^{\infty} (n+1) \frac{z^n}{\gamma^{n+2}} \\ &= \frac{1}{z^2} + \sum_{n=1}^{\infty} (n+1) G_{n+2} z^n, \end{aligned}$$

where the constants G_n, depending on the lattice Γ, are defined by the formula

$$G_n = \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\gamma^n}$$

for $n \geq 3$. By symmetry, G_n can only be nonzero for even values of n; therefore

$$P(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2} z^{2n} \tag{1.4}$$

is the Laurent expansion of $P(z)$ around the point $z = 0$.

The next lemma shows that the Weierstrass $\wp$-function is related to cubic curves.

Lemma 1.5. *The $\wp$-function satisfies the differential equation*

$$P'(z)^2 = 4P(z)^3 - 60G_4 P(z) - 140G_6.$$

Proof. From the Laurent series in (1.4), we get

$$\begin{aligned} P(z) &= z^{-2} + 3G_4 z^2 + 5G_6 z^4 + \cdots \\ P'(z) &= -2z^{-3} + 6G_4 z + 20G_6 z^3 + \cdots \\ P'(z)^2 &= 4z^{-6} - 24G_4 z^{-2} - 80G_6 + \cdots \\ 4P(z)^3 &= 4z^{-6} + 36G_4 z^{-2} + 60G_6 + \cdots \end{aligned}$$

Consequently, the function $P'(z)^2 - 4P(z)^3 + 60G_4 P(z) + 140G_6$ has no pole at $z = 0$, and because it is doubly periodic, it is therefore holomorphic, hence constant. The constant value is the value at $z = 0$, which is 0. □

If we set $g_2 = 60G_4$ and $g_3 = 140G_6$, then the differential equation takes the form $P'(z)^2 = 4P(z)^3 - g_2 P(z) - g_3$. Now consider the holomorphic mapping

$$\mathbb{C} \setminus \Gamma \to \mathbb{C}^2, \quad z \mapsto \big(P(z), P'(z)\big).$$

Its image is contained in the cubic curve with equation $y^2 = 4x^3 - g_2 x - g_3$. Let us denote by $E = \mathbb{C}/\Gamma$ the quotient, which is a compact Riemann surface of genus

1. Let $0 \in E$ be the image of the origin in $\mathbb{C}$. We get an induced holomorphic mapping

$$E \setminus \{0\} \to \mathbb{C}^2, \quad z + \Gamma \mapsto \big(P(z), P'(z)\big).$$

Since $P(z)$ has a pole of order 2 at $z = 0$, whereas $P'(z)$ has a pole of order 3, this extends to a holomorphic mapping

$$E \to \mathbb{P}^2$$

by sending the point $0 \in E$ to the point $[0, 1, 0] \in \mathbb{P}^2$. The image is now contained inside the projective cubic curve with equation

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$$