Math 589
Problem Set 7

due Monday, April 20

Most problems are from Hartshorne, *Algebraic Geometry.*

1. Use `Macaulay2` to solve this problem. Determine the number of lines on the quartic surface $x_0^4 + x_1^4 + x_2^4 + x_3^4$ in $\mathbb{P}^3$ (in characteristic 0).

2. (a) Consider a graded ring

$$S = \bigoplus_{d=0}^{\infty} S_d,$$

and for $n \geq 1$, form the subring

$$S^{(n)} = \bigoplus_{d=0}^{\infty} S_{nd}.$$

Show that $\operatorname{Proj} S \cong \operatorname{Proj} S^{(n)}$.

(b) As in class, let $\tilde{S}$ be the graded $\mathbb{Z}$-algebra obtained as the quotient of the polynomial ring $\mathbb{Z}[z_{i,j} \mid 0 \leq i, j \leq n]$ by the Segre ideal generated by all quadratic polynomials of the form $z_{i,j}z_{k,\ell} - z_{i,\ell}z_{k,j}$. Show that the ring homomorphism

$$\tilde{S} \to \mathbb{Z}[x_0, \ldots, x_n], \quad z_{i,j} \mapsto x_i x_j,$$

defines a closed embedding of $\mathbb{P}^n_{\mathbb{Z}}$ into the projective scheme $\operatorname{Proj} \tilde{S}$.

3. A morphism $f \colon X \to Y$ is *finite* if there is a covering of Y by affine open subsets $U_i = \operatorname{Spec} A_i$, such that $f^{-1}(U_i) = \operatorname{Spec} B_i$ is affine and the ring extension $A_i \to B_i$ is integral.

   (a) Show that every finite morphism is proper.

   (b) *Show that a proper morphism between affine schemes is finite.

4. Exercise II.4.7 in Hartshorne

5. Exercise II.4.8 in Hartshorne