Math 589
Problem Set 5

due Wednesday, April 1, 2026

Most problems are from Hartshorne, *Algebraic Geometry.*

1. Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The support $\operatorname{Supp} s$ is defined to be $\{ x \in U \mid s_x \neq 0 \}$, where s_x denotes the germ of s in the stalk $\mathscr{F}_x$. Show that $\operatorname{Supp} s$ is a closed subset of U.

2. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on a topological space X. For $x \in X$, denote by $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ the induced morphism on stalks. Show that the rule

$$(\operatorname{im}\varphi)(U) = \left\{ s \in \mathscr{G}(U) \,\middle|\, s_x \in \operatorname{im}\varphi_x \text{ for all } x \in U \right\}$$

defines a subsheaf of $\mathscr{G}$, and that φ is surjective if and only if $\operatorname{im}\varphi = \mathscr{G}$.

3. Let A be a ring and let $(X, \mathscr{O}_X)$ be a scheme. Given a morphism of schemes $f\colon X \to \operatorname{Spec} A$, we have an associated morphism of sheaves $f^\sharp\colon \mathscr{O}_{\operatorname{Spec} A} \to f_*\mathscr{O}_X$. Taking globel sections, we get a homomorphism $A \to \mathscr{O}_X(X)$. Thus there is a natural map

$$\alpha\colon \operatorname{Hom}_{\text{Schemes}}(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\text{Rings}}\big(A, \mathscr{O}_X(X)\big).$$

Show that α is bijective.

4. Suppose that $f\colon X \to Y$ is a morphism of finite type. Show that for every affine open subset $V \subseteq Y$, and for every affine open subset $U \subseteq f^{-1}(V)$, the ring $\mathscr{O}_X(U)$ is a finitely generated algebra over $\mathscr{O}_Y(V)$.

5. Let $\varphi\colon S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that this induces a morphism of schemes

$$f\colon \operatorname{Proj} T \to \operatorname{Proj} S,$$

and that f is a closed embedding.

6. Let A be a ring. Show that the following conditions are equivalent:

   (a) $\operatorname{Spec} A$ is disconnected.

(b) There exist nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$. Such elements are called orthogonal idempotents.

(c) A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

7. If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.