Math 589
Problem Set 4

due Monday, March 23, 2026

Some problems are from Shafarevich, *Basic Algebraic Geometry I.*

1. Let $(A, \mathfrak{m})$ be a local ring with residue field $k = A/\mathfrak{m}$. Show that $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$. (*Hint:* Argue by induction on $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$.)

2. Let $(A, \mathfrak{m})$ be a regular local ring of dimension $n = \dim A$. The goal of this exercise is to show that A is a domain.

   (a) We proved in class that there are elements $f_1, \ldots, f_n \in A$ such that $\mathfrak{m} = (f_1, \ldots, f_n)$. Show that $A/(f_1)$ is a regular local ring of dimension $n - 1$.

   (b) By induction, $A/(f_1)$ is a domain, and so (f_1) is a prime ideal. By the principal ideal theorem, it has height 1, and so there exists a prime ideal P properly contained in (f_1). Prove that $P = f_1 P$.

   (c) Use Nakayama's lemma to deduce that $P = (0)$, and hence that A is a domain.

3. Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $Z \subseteq X$ be an irreducible subvariety. Set $R = A(X)$ and $P = I(X)$. Give a geometric interpretation of the localization R_P, similar to what we did for the case when Z is a point and P a maximal ideal.

4. Determine the local ring at $(0, 0, 0)$ of the curve consisting of the three coordinate axes in $\mathbb{A}^3$.

5. Let $X \subseteq \mathbb{P}^n$ be a hypersurface defined by a homogeneous polynomial $F(x_0, \ldots, x_n)$. Show that the set of singular points of X is defined by the system of equations

$$F = \frac{\partial F}{\partial x_0} = \cdots = \frac{\partial F}{\partial x_n} = 0.$$

6. Determine the singular points of the Steiner surface in $\mathbb{P}^3$:

$$x_1^2 x_2^2 + x_2^2 x_0^2 + x_0^2 x_1^2 - x_0 x_1 x_2 x_3 = 0.$$

7. Prove that if $X = X_1 \cup X_2 \subseteq \mathbb{A}^n$ and if $x \in X_1 \cap X_2$, then

$$T_x X_1 + T_x X_2 \subseteq T_x X.$$

Does equality always hold?

8. Let $F(x_0, x_1, x_2, x_3)$ be a nontrivial homogeneous polynomial of degree 4. Prove that there is a polynomial Φ in the coefficients of F such that $\Phi(F) = 0$ is a necessary and sufficient condition for the surface $V(F) \subseteq \mathbb{P}^3$ to contain a line.