

**Math 589**  
**Problem Set 2**

due Monday, February 23, 2026

These problems are from Shafarevich, *Basic Algebraic Geometry I*, and from Artin, *Notes for a Course in Algebraic Geometry*.

1. Let  $X \subseteq \mathbb{A}^3$  be an algebraic curve, and  $x, y, z$  coordinates in  $\mathbb{A}^3$ ; suppose that  $X$  does not contain a line parallel to the  $z$ -axis. Prove that there exists a nonzero polynomial  $f(x, y)$  vanishing at all points of  $X$ . Prove that all such polynomials form a principal ideal  $(g(x, y))$ , and that the curve  $g(x, y) = 0$  in  $\mathbb{A}^2$  is the closure of the projection of  $X$  onto the  $(x, y)$ -plane parallel to the  $z$ -axis.
2. In the notation of (1), suppose that  $h(x, y, z) = g_0(x, y)z^n + \cdots + g_n(x, y)$  is the irreducible polynomials of smallest positive degree in  $z$  contained in the ideal  $I(X)$ . Prove that if  $f \in I(X)$  has degree  $m$  as a polynomial in  $z$ , then we can write  $fg_0^m = hU + v(x, y)$ , where  $v(x, y)$  is divisible by  $g(x, y)$ . Deduce that the equation  $h = g = 0$  defines a reducible curve consisting of  $X$  together with a finite number of lines parallel to the  $z$ -axis, defined by  $g_0(x, y) = g(x, y) = 0$ .
3. Use (1) and (2) to prove that any curve  $X \subseteq \mathbb{A}^3$  can be defined by 3 equations.
4. Is the morphism  $f: \mathbb{A}^1 \rightarrow X$  finite, where  $X$  is given by  $y^2 = x^3$ , and  $f(t) = (t^2, t^3)$ ?
5. Consider the curve  $X$  defined by  $y^2 = x^3 + x^2$ , and the rational function  $t = y/x$ . Show that  $t \notin A(X)$ .
6. Consider the morphism  $f: \mathbb{A}^2 \rightarrow \mathbb{A}^3$  defined by  $f(x, y) = (x, xy, xyz)$ . Find the image  $f(\mathbb{A}^2)$ . Is it open in  $\mathbb{A}^3$ ? Is it dense? Is it closed?
7. Let  $Y$  and  $Z$  be closed subsets of an affine variety  $X$ . Suppose that no irreducible component of  $Z$  is contained in  $Y$ . Prove that there is a regular function  $f \in A(X)$  that vanishes on  $Y$  and is not identically zero on any component of  $Z$ .
8. Let  $X \subseteq \mathbb{A}^3$  be the curve defined by the equations  $z^3 = x^2$  and  $yz^2 + z + y = 0$ . Determine the equation of the projection of  $C$  to the  $(x, y)$ -plane.