Irreducible finite-dimensional representations of semisimple Lie algebras

MAT 552

May 6, 2020

Brief review: weight spaces

Let V be a representation of $\mathfrak{g}$. The subspace

$$V[\lambda] = \left\{ v \in V \;\middle|\; h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h} \right\}$$

is called the weight space of weight $\lambda \in \mathfrak{h}^*$.

Notation: $P(V) = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; V[\lambda] \neq 0 \right\}$

When V is finite-dimensional, we proved:

- The weight decomposition $V = \bigoplus_{\lambda \in P(V)} V[\lambda]$.
- $\dim V[\lambda] = \dim V[w(\lambda)]$ for every $w \in W$.
- $P(V)$ is a subset of the weight lattice

$$P = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \alpha \in R \right\}.$$

Example: weight lattice in type G_2

Dynkin diagram of type G_2: 

There are two simple roots α_1 and α_2 with

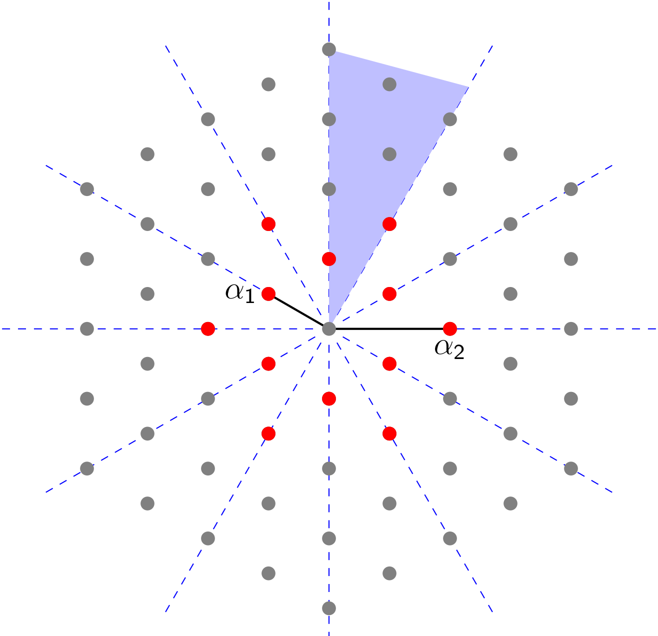
$$\frac{2(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = -3 \quad \text{and} \quad \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = -1.$$

The weight lattice P is the set of vectors $x\alpha_1 + y\alpha_2$ with

$$\frac{2x(\alpha_1, \alpha_1) + 2y(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = 2x - 3y \in \mathbb{Z}$$
$$\frac{2x(\alpha_2, \alpha_1) + 2y(\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = 2y - x \in \mathbb{Z}$$

In this case, it follows that $x, y \in \mathbb{Z}$, and so P is exactly the lattice generated by the two simple roots α_1 and α_2.

Example: weight lattice of type G_2



Brief review: highest weight representations

A nonzero representation V of $\mathfrak{g}$ is called a highest weight representation if it is generated by a vector $v \in V[\lambda]$ with

$$x \cdot v = 0 \quad \text{for all } x \in \mathfrak{n}_+.$$

The decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ comes from a choice of polarization $R = R_+ \sqcup R_-$ of the root system.

Last time, we proved:

- Every irreducible finite-dimensional representation is a highest weight representation.
- Every highest weight representation of highest weight λ is isomorphic to a quotient of the Verma module M_λ.
- Every highest weight representation admits a weight decomposition with finite-dimensional weight spaces.
- The highest weight λ is unique, and $\dim V[\lambda] = 1$.

Irreducible finite-dimensional representations

Goal: Classify irreducible finite-dimensional representations.

Every such representation is highest weight representation, hence a quotient of some M_λ. Two natural questions:

1. Which quotients of M_λ are irreducible?
2. Which of these are finite-dimensional?

The answer to the first question is easy.

Theorem

For each $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has a unique quotient L_λ that is irreducible as a representation of $\mathfrak{g}$.

This says that, up to isomorphism, there is a unique irreducible highest weight representation of highest weight λ.

Irreducible highest weight representations

Theorem

For each $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has a unique quotient L_λ that is irreducible as a representation of $\mathfrak{g}$.

Proof:

- The quotient M_λ/W is nonzero and irreducible $\iff W$ is maximal among proper submodules of M_λ.
- Every submodule W admits a weight decomposition. (Reason: M_λ admits a weight decomposition.)
- If $W \neq M_\lambda$, then $W[\lambda] = 0$. (Reason: $M_\lambda[\lambda]$ generates M_λ, and $\dim M_\lambda[\lambda] = 1$)
- Let W_λ be the sum over all W such that $W[\lambda] = 0$.
- Clearly W_λ is the unique maximal proper submodule.
- Therefore $L_\lambda = M_\lambda/W_\lambda$ does the job. $\square$

Irreducible finite-dimensional representations

Goal: Classify irreducible finite-dimensional representations.

Every such representation is isomorphic to L_λ for a unique weight $\lambda \in \mathfrak{h}^*$. It remains to figure out when $\dim L_\lambda < \infty$.

The answer to this question is less easy, but still very pretty.

Main theorem

L_λ is finite-dimensional $\iff \lambda \in P_+$

Definition: A weight $\lambda \in \mathfrak{h}^*$ is called dominant integral if

$$\frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{N} \quad \text{for all } \alpha \in R_+.$$

The set of dominant integral weights is denoted P_+.

Dominant integral weights

Definition: A weight $\lambda \in \mathfrak{h}^*$ is called dominant integral if

$$\frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{N} \quad \text{for all } \alpha \in R_+.$$

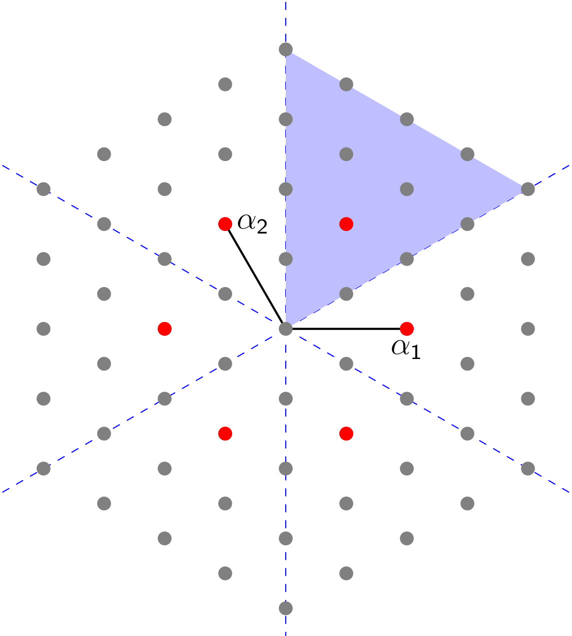
The set of dominant integral weights is denoted P_+.

- By definition, P_+ is contained in the weight lattice

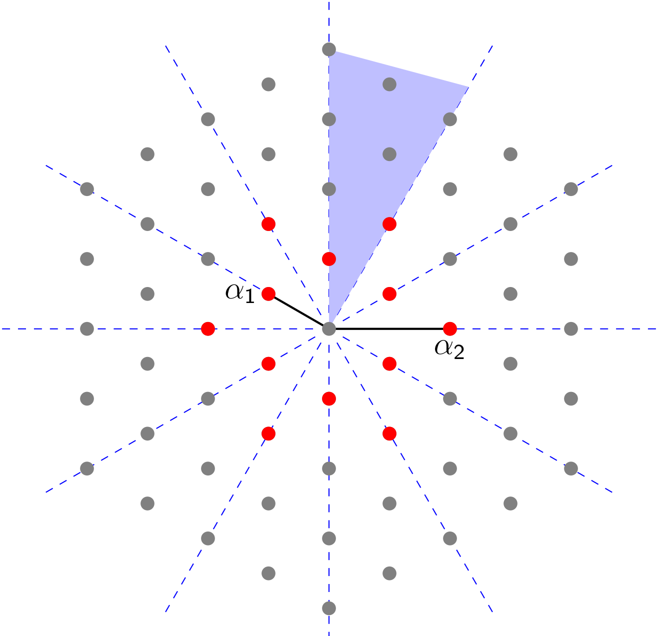
$$P = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{Z} \text{ for all } \alpha \in R \right\}.$$

- It is enough to check the condition for all simple roots.
- $P_+ = P \cap \overline{C_+}$ is the intersection of the weight lattice P with the closure of the positive Weyl chamber C_+.

Example: $\mathfrak{sl}(3, \mathbb{C})$



Example: Lie algebra of type G_2



Proof of main theorem

Main theorem

L_λ is finite-dimensional $\iff \lambda \in P_+$

Proof: I will only prove the easy direction.

- Suppose that $\dim L_\lambda < \infty$.
- Let $\alpha \in R_+$ be a positive root, $\mathfrak{sl}(2,\mathbb{C})_\alpha = \langle e_\alpha, f_\alpha, h_\alpha \rangle$.
- The highest weight vector $v_\lambda \in L_\lambda$ satisfies

$$e_\alpha v_\lambda = 0 \quad \text{and} \quad h_\alpha v_\lambda = \lambda(h_\alpha) v_\lambda = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} v_\lambda.$$

- But in a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, the highest weight is a nonnegative integer.
- Therefore $2(\alpha,\lambda)/(\alpha,\alpha) \in \mathbb{N}$, hence $\lambda \in P_+$. $\square$

Irreducible finite-dimensional representations

The main theorem solves our problem (at least in theory):

1. For every $\lambda \in P_+$, the representation L_λ is irreducible and finite-dimensional.
2. These representations are pairwise non-isomorphic.
3. Every irreducible finite-dimensional representation is isomorphic to a unique L_λ.

In practice, there are usually better ways to get your hands on the irreducible finite-dimensional representations.

Computing dimensions

Define $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$. This is a special element in E.

Theorem

One has $\dim L_\lambda = \prod_{\alpha \in R_+} \left(1 + \frac{(\alpha, \lambda)}{(\alpha, \rho)}\right)$ *for every* $\lambda \in P_+$.

This is a consequence of the Weyl character formula.

For $\lambda \in P_+$, it gives a (big) formula for the character

$$\mathrm{ch}(L_\lambda) = \sum_\mu \dim L_\lambda[\mu] \cdot e^\mu \in \mathbb{C}[P],$$

involving the set of positive roots R_+, the Weyl group W, and the special element ρ.

Example: $\mathfrak{sl}(3,\mathbb{C})$

For the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$, we had

$$E \cong \mathbb{R}^3/\mathbb{R}(e_1+e_2+e_3) \cong \left\{ x \in \mathbb{R}^3 \;\middle|\; x_1+x_2+x_3=0 \right\}.$$

The simple roots are $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$. Hence

$$\frac{2(\alpha_1,\alpha_2)}{(\alpha_1,\alpha_1)} = \frac{2(\alpha_1,\alpha_2)}{(\alpha_2,\alpha_2)} = -1.$$

A weight $\lambda = x\alpha_1 + y\alpha_2$ belongs to P_+ iff

$$a = \frac{2x(\alpha_1,\alpha_1)+2y(\alpha_1,\alpha_2)}{(\alpha_1,\alpha_1)} = 2x - y \in \mathbb{N}$$
$$b = \frac{2x(\alpha_2,\alpha_1)+2y(\alpha_2,\alpha_2)}{(\alpha_2,\alpha_2)} = 2y - x \in \mathbb{N}$$

Solving for x, y, we get $x = \frac{1}{3}(2a+b)$ and $y = \frac{1}{3}(a+2b)$.

Example: $\mathfrak{sl}(3, \mathbb{C})$

Dominant integral weights are therefore of the form

$$\lambda = \frac{2a+b}{3}\alpha_1 + \frac{a+2b}{3}\alpha_2$$

for a pair of natural numbers $a, b \in \mathbb{N}$.

These are in one-to-one correspondence with irreducible finite-dimensional representations.

The dimension formula (with $\rho = \alpha_1 + \alpha_2$) gives the dimension of the corresponding representation as

$$\dim L_\lambda = (1+a)(1+b)\left(1 + \frac{a+b}{2}\right).$$

Example: $\mathfrak{sl}(3, \mathbb{C})$

The standard representation on $\mathbb{C}^3$ has weights

$$\begin{aligned} e_1 &\equiv \tfrac{1}{3}(2e_1 - e_2 - e_3) = \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2 = \lambda, \\ e_2 &= e_1 - (e_1 - e_2) \equiv \lambda - \alpha_1, \\ e_3 &= e_2 - (e_2 - e_3) \equiv \lambda - \alpha_1 - \alpha_2. \end{aligned}$$

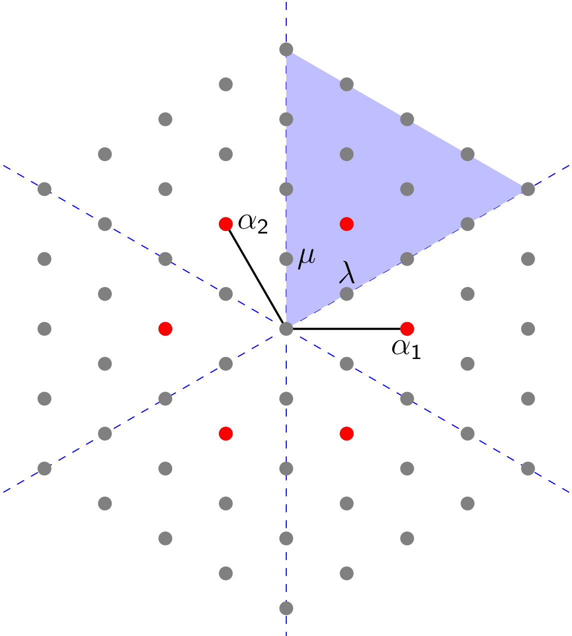
It corresponds to $(a, b) = (1, 0)$.

The representation $\wedge^2\mathbb{C}^3$ has weights

$$\begin{aligned} e_1 + e_2 &\equiv \tfrac{1}{3}(e_1 + e_2 - 2e_3) = \tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2 = \mu, \\ e_1 + e_3 &= (e_1 + e_2) - (e_2 - e_3) \equiv \mu - \alpha_2, \\ e_2 + e_3 &= (e_1 + e_3) - (e_1 - e_2) \equiv \mu - \alpha_1 - \alpha_2. \end{aligned}$$

It corresponds to $(a, b) = (0, 1)$.

Example: $\mathfrak{sl}(3, \mathbb{C})$



Example: Lie algebra of type G_2