Representation theory of semisimple Lie algebras

MAT 552

May 4, 2020

Brief review: representations of $\mathfrak{sl}(2, \mathbb{C})$

A few weeks ago, we proved the following results:

- Every finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ is completely reducible.
- Up to isomorphism, there is a unique irreducible representation of every finite dimension.
- Every finite-dimensional representation V decomposes as

$$V = \bigoplus_{n \in \mathbb{Z}} V[n]$$

  into weight spaces $V[n] = \left\{ v \in V \;\middle|\; h \cdot v = nv \right\}$.
- One has the symmetry $\dim V[n] = \dim V[-n]$.

Brief review: representations of $\mathfrak{sl}(2, \mathbb{C})$

The irreducible representation of dimension $n+1$ is

$$V_n = \langle v_0, v_1, \ldots, v_n \rangle,$$

with the action by $\mathfrak{sl}(2, \mathbb{C})$ given by the following formulas:

$$\begin{aligned} hv_k &= (n-2k)v_k \\ fv_k &= (k+1)v_{k+1} \\ ev_k &= (n-k+1)v_{k-1} \end{aligned}$$

The vector v_0 is called a highest weight vector because

$$hv_0 = nv_0 \quad \text{and} \quad ev_0 = 0.$$

Brief review: representations of $\mathfrak{sl}(2, \mathbb{C})$

In fact, for any $\lambda \in \mathbb{C}$, one can define a representation

$$M_\lambda = \langle v_0, v_1, \ldots, v_k, \ldots \rangle$$

with the action by $\mathfrak{sl}(2, \mathbb{C})$ given by the following formulas:

$$\begin{aligned} hv_k &= (\lambda - 2k)v_k \\ fv_k &= (k+1)v_{k+1} \\ ev_k &= (\lambda - k + 1)v_{k-1} \end{aligned}$$

The vector v_0 is still a highest weight vector.

- When $\lambda \notin \mathbb{N}$, the representation M_λ is irreducible.
- When $\lambda \in \mathbb{N}$, the subspace

  $$W = \langle v_{\lambda+1}, v_{\lambda+2}, \ldots \rangle$$

  is a subrepresentation, and $V_\lambda \cong M_\lambda / W$.

Representations of semisimple complex Lie algebras

Goal: Generalize this to all semisimple complex Lie algebras.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra.

- We know that every finite-dimensional representation is completely reducible.
- We will therefore focus on irreducible representations.

The main tool is the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}$ is a Cartan subalgebra.

Weight spaces

As in the example of $\mathfrak{sl}(2, \mathbb{C})$, the key idea is to decompose representations into eigenspaces for the Cartan subalgebra $\mathfrak{h}$.

Let V be a representation of $\mathfrak{g}$.

We say that a vector $v \in V$ has weight $\lambda \in \mathfrak{h}^*$ if

$$h \cdot v = \lambda(h)v \quad \text{for all } h \in \mathfrak{h}.$$

The subspace of vectors of a given weight $\lambda \in \mathfrak{h}^*$ is denoted

$$V[\lambda] = \left\{ v \in V \;\middle|\; h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h} \right\}$$

and is called the weight space of weight λ.

If $V[\lambda]$ is nonzero, we say that λ is a weight of V.

Notation: $P(V) = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; V[\lambda] \neq 0 \right\}$

Weight decomposition and weight lattice

Theorem

Let V be a finite-dimensional representation of $\mathfrak{g}$.

1. *One has a weight decomposition*
$$V = \bigoplus_{\lambda \in P(V)} V[\lambda].$$
2. *For any root $\alpha \in R$, one has $\mathfrak{g}_\alpha \cdot V[\lambda] \subseteq V[\lambda + \alpha]$.*
3. *$P(V)$ is always a subset of the weight lattice*
$$P = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \alpha \in R \right\} \subseteq E.$$

Example

For the adjoint representation of $\mathfrak{g}$, the weight decomposition is the root decomposition, and $P(\mathfrak{g}) = R \cup \{0\}$.

Weight decomposition and weight lattice

Proof:

- Let $\alpha \in R$ be a root, and $h_\alpha \in \mathfrak{h}$ the element with

$$\lambda(h_\alpha) = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \quad \text{for all } \lambda \in \mathfrak{h}^*.$$

- Consider V as a representation of the subalgebra

$$\mathfrak{sl}(2,\mathbb{C})_\alpha = \mathfrak{g}_{-\alpha} \oplus \mathbb{C}h_\alpha \oplus \mathfrak{g}_\alpha.$$

- From the representation theory of $\mathfrak{sl}(2,\mathbb{C})$, we know that $h_\alpha \in \mathrm{End}(V)$ is diagonalizable, with integer eigenvalues.
- Since $\mathfrak{h}$ is commutative, we obtain

$$V = \bigoplus_{\lambda \in P(V)} V[\lambda].$$

Weight decomposition and weight lattice

- Let $\alpha \in R$ and $\lambda \in P(V)$. For any $v \in V[\lambda]$, we have

$$h_\alpha \cdot v = \lambda(h_\alpha)v = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}v.$$

- Since h_α has integer eigenvalues, we get

$$\frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{Z}$$

  for every root $\alpha \in R$, hence $P(V) \subseteq P$.
- If $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ are the simple roots, then

$$P = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)} \in \mathbb{Z} \text{ for } i = 1, \ldots, n \right\}.$$

- Therefore P is indeed a lattice in E. $\square$

Example: weight lattice of $\mathfrak{sl}(3, \mathbb{C})$

For the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$, we had

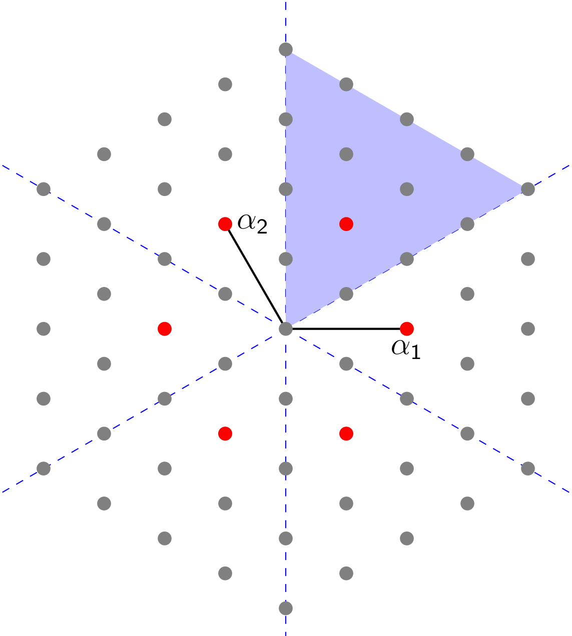
$$\frac{2(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = -1.$$

The weight lattice P is the set of vectors $x\alpha_1 + y\alpha_2$ with

$$\frac{2x(\alpha_1, \alpha_1) + 2y(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = 2x - y \in \mathbb{Z}$$
$$\frac{2x(\alpha_2, \alpha_1) + 2y(\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = 2y - x \in \mathbb{Z}$$

It follows that $x, y \in \frac{1}{3}\mathbb{Z}$.

Example: weight lattice of $\mathfrak{sl}(3, \mathbb{C})$



Weight spaces and the Weyl group

Theorem

Let V be a finite-dimensional representation of $\mathfrak{g}$. Then

$$\dim V[\lambda] = \dim V[w(\lambda)]$$

for every element $w \in W$ of the Weyl group.

Example

In the case of $\mathfrak{sl}(2, \mathbb{C})$, the Weyl group has two elements that act as ± 1 on $E \cong \mathbb{R}$; this explains why

$$\dim V[n] = \dim V[-n].$$

Weight spaces and the Weyl group

Proof:

- W is generated by the simple reflections $s_1, \ldots, s_n$, corresponding to the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$.
- We may therefore assume that $w = s_i$. Now

  $$w(\lambda) = \lambda - n\alpha_i,$$

  where $n = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i) \in \mathbb{Z}$.
- Consider V as a representation of $\mathfrak{sl}(2, \mathbb{C})_{\alpha_i} = \langle e_i, f_i, h_i \rangle$.
- As a representation of $\mathfrak{sl}(2, \mathbb{C})$, the subspace $V[\lambda]$ has weight n, and the subspace $V[\lambda - n\alpha_i]$ has weight $-n$.
- From the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, we know that

  $$f_i^n\colon V[\lambda] \to V[\lambda - n\alpha_i] \quad \text{and} \quad e_i^n\colon V[\lambda - n\alpha_i] \to V[\lambda]$$

  are isomorphisms.
- This gives $\dim V[\lambda] = \dim V[w(\lambda)]$. $\square$

Highest weight representations

Recall that a choice of polarization $R = R_+ \sqcup R_-$ of the root system gives us a decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

into $\mathfrak{h}$ and the two nilpotent subalgebras $\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$.

Definition

A nonzero representation V of $\mathfrak{g}$ is called a highest weight representation if it is generated by a vector $v \in V[\lambda]$ with

$$x \cdot v = 0 \quad \text{for all } x \in \mathfrak{n}_+.$$

In this case, v is called the highest weight vector.

Highest weight representations

Theorem

Every irreducible finite-dimensional representation of $\mathfrak{g}$ is a highest weight representation.

Proof:

- Choose $\lambda \in P(V)$ such that $\lambda + \alpha \notin P(V)$ for all $\alpha \in R_+$.
- This exists: take $h \in \mathfrak{h}$ such that $\alpha(h) > 0$ for all $\alpha \in R_+$, and then choose $\lambda \in P(V)$ such that $\lambda(h)$ is maximal.
- Let $v \in V[\lambda]$ be a nonzero vector.
- Since $\lambda + \alpha \notin P(V)$, we have $x \cdot v = 0$ for every $x \in \mathfrak{n}_+$.
- The subrepresentation generated by v is a highest weight representation; by irreducibility, it must be all of V. $\square$

Verma modules

Goal: Construct all highest weight representations.

Let V be a highest weight representation, $v \in V[\lambda]$. Then

$$h \cdot v = \lambda(h)v \quad \text{for all } h \in \mathfrak{h},$$
$$x \cdot v = 0 \quad \text{for all } x \in \mathfrak{n}_+.$$

Consider the "universal highest weight representation" M_λ, generated by a vector v_λ that satisfies the two relations above.

Using the universal enveloping algebra, we can define

$$M_\lambda = U\mathfrak{g}/I_\lambda,$$

where I_λ is the left ideal generated by the elements $x \in \mathfrak{n}_+$ and $h - \lambda(h)$, for $h \in \mathfrak{h}$. This is called the Verma module.

Verma modules

Recall that representations of the Lie algebra $\mathfrak{g}$ are the same thing as left modules over the universal enveloping algebra $U\mathfrak{g}$.

- The Verma module M_λ is a representation of $\mathfrak{g}$.
- We will see in a moment that $\dim M_\lambda = \infty$.

Verma modules are universal in the following sense.

Lemma

If V is a highest weight representation of highest weight λ, then one has

$$V \cong M_\lambda / W$$

for some submodule $W \subseteq M_\lambda$.

Understanding highest weight representations is therefore equivalent to understanding submodules of Verma modules.

Verma modules

Here is an alternative construction of M_λ. The subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$$

is solvable; it is called the Borel subalgebra.

The relations above give a representation $\rho\colon \mathfrak{b} \to \mathrm{End}(\mathbb{C})$, with

$$\begin{aligned} \rho(h) &= \lambda(h) \quad \text{for all } h \in \mathfrak{h}, \\ \rho(x) &= 0 \quad \text{for all } x \in \mathfrak{n}_+. \end{aligned}$$

Denote this representation by the symbol $\mathbb{C}_\lambda$. Then

$$M_\lambda \cong U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda,$$

where the tensor product is over the subalgebra $U\mathfrak{b} \subseteq U\mathfrak{g}$.

Verma modules

Theorem

Let $\lambda \in \mathfrak{h}^*$, and let M_λ be the Verma module.

1. *The multiplication map*

$$U\mathfrak{n}_- \to M_\lambda, \quad u \mapsto uv_\lambda,$$

   is an isomorphism of vector spaces.

2. *M_λ has a weight decomposition*

$$M_\lambda = \bigoplus_\mu M_\lambda[\mu]$$

   with finite-dimensional weight spaces $M_\lambda[\mu]$.

3. $P(M_\lambda) = \lambda - Q_+ = \lambda - \left\{ \sum_{i=1}^{n} k_i \alpha_i \;\middle|\; k_1, \ldots, k_n \in \mathbb{N} \right\}$
4. *One has $M_\lambda[\lambda] = 1$.*

Verma modules

Proof:

- Poincaré-Birkhoff-Witt theorem: $\mathfrak{g}$ embeds into $U\mathfrak{g}$, and if $x_1, \ldots, x_d$ is an ordered basis for $\mathfrak{g}$, then the monomials $x_1^{a_1} \cdots x_d^{a_d}$ form a basis for $U\mathfrak{g}$.
- Since $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}$, it follows that

$$U\mathfrak{g} \cong U\mathfrak{n}_- \otimes_{\mathbb{C}} U\mathfrak{b}$$

  as left $U\mathfrak{n}_-$-modules (and right $U\mathfrak{b}$-modules).
- Using the alternative description of the Verma module,

$$\begin{aligned} M_\lambda \cong U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda &\cong \left(U\mathfrak{n}_- \otimes_{\mathbb{C}} U\mathfrak{b}\right) \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda \\ &\cong U\mathfrak{n}_- \otimes_{\mathbb{C}} \mathbb{C}_\lambda. \end{aligned}$$

- This implies (1); the other claims follow easily. $\square$

Highest weight representations

Corollary

Let V be a highest weight representation of highest weight λ.

1. *The multiplication map*

$$U\mathfrak{n}_- \to V, \quad u \mapsto uv_\lambda,$$

   is surjective.

2. *V admits a weight decomposition*

$$V = \bigoplus_{\mu \in \lambda - Q_+} V[\mu]$$

   with finite-dimensional weight spaces $V[\mu]$.

3. *One has* $\dim V[\lambda] = 1$.

4. *The highest weight is unique, and the highest weight vector is unique up to rescaling.*

Highest weight representations

Proof:

- We have $V \cong M_\lambda / W$ for a submodule $W \subseteq M_\lambda$.
- Therefore (1) and (2) follow from the theorem about M_λ.
- For the same reason, $\dim V[\lambda] \leq 1$.
- Since V is a highest weight representation, it is a generated by a highest weight vector $v \in V[\lambda]$.
- Therefore $V[\lambda] = \mathbb{C}v$, and v is unique up to scaling.
- If λ' was another highest weight, then

$$\lambda' \in \lambda - Q_+ \quad \text{and} \quad \lambda \in \lambda' - Q_+.$$

- Therefore $\lambda' - \lambda$ and $\lambda - \lambda'$ belong to

$$Q_+ = \left\{ \sum_{i=1}^{n} k_i \alpha_i \;\middle|\; k_1, \ldots, k_n \in \mathbb{N} \right\},$$

and so $\lambda' = \lambda$. $\square$