Representations of $\mathfrak{sl}(2, \mathbb{C})$ and semisimple/nilpotent elements

MAT 552

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Topic 1: Representation theory of $\mathfrak{sl}(2,\mathbb{C})$

In the first half of today's class, we are going to describe all finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$:

- an interesting example
- needed for the study of arbitrary semisimple Lie algebras
- shows up in many other parts of mathematics

Recall that $\mathfrak{sl}(2,\mathbb{C})$ is generated by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They satisfy the relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Complete reducibility

Recall that $\mathfrak{sl}(2, \mathbb{C})$ is a simple Lie algebra, hence semisimple.

By the theorem from last time, every finite-dimensional representation is completely reducible ($=$ a direct sum of irreducible representations).

So we only need to understand the irreducible representations.

Example

The standard representation on $\mathbb{C}^2$ is irreducible.
The adjoint representation is irreducible (of dimension 3).

We will see that, for every $n \geq 0$, there is a unique irreducible representation of dimension $n + 1$ (up to isomorphism).

Irreducible representations

Let V be a finite-dimensional irreducible representation.

The key idea is to look at the eigenspaces of $h \in \mathrm{End}(V)$.
For any $\lambda \in \mathbb{C}$, define

$$V[\lambda] = \left\{ v \in V \;\middle|\; hv = \lambda v \right\}.$$

Lemma

We have $e \cdot V[\lambda] \subseteq V[\lambda + 2]$ and $f \cdot V[\lambda] \subseteq V[\lambda - 2]$.

Indeed, suppose that $hv = \lambda v$. Then

$$\begin{aligned} h(ev) &= e(hv) + [h, e]v = e(\lambda v) + 2ev = (\lambda + 2)ev, \\ h(fv) &= f(hv) + [h, f]v = f(\lambda v) - 2fv = (\lambda - 2)fv. \end{aligned}$$

Irreducible representations

Let V be a finite-dimensional irreducible representation.

Let $v \in V[\lambda]$ be a nonzero eigenvector with $\mathrm{Re}\,\lambda$ maximal.
Since $ev \in V[\lambda + 2]$, we must have $ev = 0$.

Consider the sequence of vectors

$$v_0 = v, \quad v_1 = fv, \quad v_2 = \frac{f^2 v}{2!}, \quad v_3 = \frac{f^3 v}{3!}, \ldots$$

Clearly $v_k \in V[\lambda - 2k]$.

Lemma

We have $fv_k = (k+1)v_{k+1}$ and $ev_k = (\lambda - k + 1)v_{k-1}$.

Here $v_{-1} = 0$ for convenience.

Proof of the lemma

Lemma

We have $fv_k = (k+1)v_{k+1}$ *and* $ev_k = (\lambda - k + 1)v_{k-1}$.

The first half is clear:

$$fv_k = \frac{f^{k+1}v}{k!} = (k+1)\frac{f^{k+1}v}{(k+1)!} = (k+1)v_{k+1}$$

The second half is proved by induction on $k \geq 0$:

$$ev_{k+1} = \frac{efv_k}{k+1} = \frac{fev_k + [e,f]v_k}{k+1} = \frac{f(ev_k) + hv_k}{k+1}$$

By induction, $ev_k = (\lambda - k + 1)v_{k-1}$; also $hv_k = (\lambda - 2k)v_k$.

Proof of lemma

Lemma

We have $fv_k = (k+1)v_{k+1}$ *and* $ev_k = (\lambda - k + 1)v_{k-1}$.

Therefore

$$\begin{aligned} ev_{k+1} &= \frac{(\lambda - k + 1)fv_{k-1} + (\lambda - 2k)v_k}{k+1} \\ &= \frac{(\lambda - k + 1)kv_k + (\lambda - 2k)v_k}{k+1} \\ &= \frac{\lambda(k+1) - k(k+1)}{k+1} v_k = (\lambda - k)v_k, \end{aligned}$$

as required. $\square$

Irreducible representations

Since $v_k \in V[\lambda - 2k]$, the vectors v_k are linearly independent. But V is finite-dimensional, and so $v_k = 0$ for $k \gg 0$.

Let $n \geq 0$ be maximal with $v_n \neq 0$ (and $v_{n+1} = 0$). Then

$$0 = ev_{n+1} = (\lambda - n)v_n$$

implies that $\lambda = n$. In particular, λ is always an integer.

The formulas in the lemma show that

$$\langle v_0, v_1, \ldots, v_n \rangle \subseteq V$$

is a subrepresentation, hence equal to V (by irreducibility).

In particular, $\dim V = n + 1$.

Irreducible representations

Theorem

Up to isomorphism, $\mathfrak{sl}(2,\mathbb{C})$ has a unique irreducible representation of dimension $n+1$, for every $n \geq 0$.

Concretely, this representation is given by

$$V = \langle v_0, v_1, \ldots, v_n \rangle,$$

with the action by $\mathfrak{sl}(2,\mathbb{C})$ defined by the following rule:

$$\begin{aligned} hv_k &= (n-2k)v_k \\ fv_k &= (k+1)v_{k+1} \\ ev_k &= (n-k+1)v_{k-1} \end{aligned}$$

The vector v_0 is called a vector of highest weight.

Irreducible representations

Pictorially (with the weights in red):

$$\begin{tikzcd}
v_0 \arrow[r, bend left, "f"] \arrow[d, dash] & v_1 \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] \arrow[d, dash] & v_2 \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] \arrow[d, dash] & \cdots \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] & v_{n-1} \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] \arrow[d, dash] & v_n \arrow[l, bend left, "e"] \arrow[d, dash] \\
n & n-2 & n-4 & & -n+2 & -n
\end{tikzcd}$$

Note that

$$V = V[n] \oplus V[n-2] \oplus \cdots \oplus V[-n+2] \oplus V[-n].$$

Example

For $n = 1$, we get the standard representation on $\mathbb{C}^2$.
For $n = 2$, we get the adjoint representation on $\mathfrak{sl}(2, \mathbb{C})$.

Finite-dimensional representations

Now let V be an arbitrary finite-dimensional representation.

- V is a direct sum of irreducible representations.
- The eigenvalues of h are called the weights of the representation. They are all integers.
- We have a weight decomposition

  $$V = \bigoplus_{n \in \mathbb{Z}} V[n];$$

  e increases weights by 2, and f decreases weights by 2.
- For each $n \geq 1$, the endomorphisms

  $$f^n \colon V[n] \to V[-n] \quad \text{and} \quad e^n \colon V[-n] \to V[n]$$

  are isomorphisms. Hence $\dim V[n] = \dim V[-n]$.

Topic 2: Semisimple/nilpotent elements

We want to understand the structure of arbitrary semisimple Lie algebras (and, ultimately, classify them).

- We should use the insights we gained from $\mathfrak{sl}(2, \mathbb{C})$.
- The element $h \in \mathfrak{sl}(2, \mathbb{C})$ is special because of the two relations $[h, e] = 2e$ and $[h, f] = -2f$.
- These are saying that $\operatorname{ad} h$ is diagonal in the basis e, h, f.

Definition

Let $\mathfrak{g}$ be a Lie algebra. An element $x \in \mathfrak{g}$ is called

- semisimple if $\operatorname{ad} x \in \operatorname{End}(\mathfrak{g})$ is semisimple,
- nilpotent if $\operatorname{ad} x \in \operatorname{End}(\mathfrak{g})$ is nilpotent.

Semisimple/nilpotent elements

Definition

Let $\mathfrak{g}$ be a Lie algebra. An element $x \in \mathfrak{g}$ is called

- semisimple if $\operatorname{ad} x \in \operatorname{End}(\mathfrak{g})$ is semisimple,
- nilpotent if $\operatorname{ad} x \in \operatorname{End}(\mathfrak{g})$ is nilpotent.

Example

In $\mathfrak{sl}(2, \mathbb{C})$, the element h is semisimple, and e, f are nilpotent.

Example

A matrix $x \in \mathfrak{gl}(n, \mathbb{C})$ is semisimple iff it is diagonalizable.

Generalized Jordan decomposition

Now let $\mathfrak{g}$ be a semisimple complex Lie algebra.

Theorem

Every $x \in \mathfrak{g}$ can be uniquely written as

$$x = x_s + x_n,$$

where x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$. Moreover, if $[x, y] = 0$ for some $y \in \mathfrak{g}$, then also $[x_s, y] = 0$.

Uniqueness is easy:

- $x = x_s + x_n$ implies that $\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$.
- $\operatorname{ad} x_s$ is semisimple, $\operatorname{ad} x_n$ is nilpotent.
- $[\operatorname{ad} x_s, \operatorname{ad} x_n] = \operatorname{ad}[x_s, x_n] = 0$, so they commute.
- The Jordan decomposition of $\operatorname{ad} x$ is unique (and $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ is injective).

Generalized Jordan decomposition

Now we prove existence. Fix an element $x \in \mathfrak{g}$. Let

$$\operatorname{ad} x = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$$

be the Jordan decomposition of $\operatorname{ad} x \in \operatorname{End}(\mathfrak{g})$.

- Initially, $(\operatorname{ad} x)_s$ and $(\operatorname{ad} x)_n$ are just endomorphisms of $\mathfrak{g}$.
- So the point is to show that

  $$(\operatorname{ad} x)_s = \operatorname{ad} x_s \quad \text{and} \quad (\operatorname{ad} x)_n = \operatorname{ad} x_n$$

  for two elements $x_s, x_n \in \mathfrak{g}$.

Let $\mathfrak{g}_\lambda$ be the λ-eigenspace of $(\operatorname{ad} x)_s$. Then

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_\lambda,$$

and $(\operatorname{ad} x - \lambda \operatorname{id})^n$ acts trivially on $\mathfrak{g}_\lambda$ for $n \geq \dim \mathfrak{g}_\lambda$.

Generalized Jordan decomposition

Lemma

We have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$.

The Jacobi identity gives

$$(\operatorname{ad} x - \lambda - \mu)[y, z] = [(\operatorname{ad} x - \lambda)y, z] + [y, (\operatorname{ad} x - \mu)z].$$

This implies, by induction, that

$$(\operatorname{ad} x - \lambda - \mu)^n[y, z] = \sum_{k=0}^{n} \binom{n}{k} [(\operatorname{ad} x - \lambda)^k y, (\operatorname{ad} x - \mu)^{n-k} z].$$

If $y \in \mathfrak{g}_\lambda$ and $z \in \mathfrak{g}_\mu$, then the right-hand side vanishes once $n \geq \dim \mathfrak{g}_\lambda + \dim \mathfrak{g}_\mu - 1$. Therefore $[y, z] \in \mathfrak{g}_{\lambda+\mu}$. $\square$

Generalized Jordan decomposition

Recall the Jordan decomposition $\operatorname{ad} x = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$.

The lemma shows that $(\operatorname{ad} x)_s \in \operatorname{End}(\mathfrak{g})$ is a derivation:

$$(\operatorname{ad} x)_s\,[y, z] = [(\operatorname{ad} x)_s\, y, z] + [y, (\operatorname{ad} x)_s\, z]$$

It is enough to check this for $y \in \mathfrak{g}_\lambda$ and $z \in \mathfrak{g}_\mu$. Then

$$\begin{aligned} [(\operatorname{ad} x)_s\, y, z] + [y, (\operatorname{ad} x)_s\, z] &= [\lambda y, z] + [y, \mu z], \\ (\operatorname{ad} x)_s\,[y, z] &= (\lambda + \mu)[y, z], \end{aligned}$$

because $[y, z] \in \mathfrak{g}_{\lambda+\mu}$ by the lemma.

We proved earlier that every derivation of a semisimple Lie algebra is an inner derivation: $\operatorname{Der}(\mathfrak{g}) = \operatorname{ad}(\mathfrak{g})$

Generalized Jordan decomposition

Recall the Jordan decomposition $\operatorname{ad} x = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$.

Conclusion:

- There is a unique $x_s \in \mathfrak{g}$ with $(\operatorname{ad} x)_s = \operatorname{ad} x_s$.
- Since $(\operatorname{ad} x)_s$ is semisimple, x_s is a semisimple element.
- It follows that $(\operatorname{ad} x)_n = \operatorname{ad} x_n$, where $x_n = x - x_s$.
- Since $(\operatorname{ad} x)_n$ is nilpotent, x_n is a nilpotent element.
- From $\operatorname{ad}[x_s, x_n] = [\operatorname{ad} x_s, \operatorname{ad} x_n] = 0$, we get $[x_s, x_n] = 0$.

Suppose that $[x, y] = 0$ for some $y \in \mathfrak{g}$.

- We have $\operatorname{ad} x.y = 0$.
- Therefore $\operatorname{ad} x_s.y = (\operatorname{ad} x)_s\, y = 0$.
- This means that $[x_s, y] = 0$. $\square$

Existence of semisimple elements

One consequence of the generalized Jordan decomposition is that every semisimple complex Lie algebra contains nonzero semisimple elements:

- Suppose that every semisimple element $x \in \mathfrak{g}$ is zero.
- By the theorem, every $x \in \mathfrak{g}$ is nilpotent.
- Consequently, $\operatorname{ad} x$ is nilpotent for every $x \in \mathfrak{g}$.
- By Engel's theorem, $\mathfrak{g}$ is nilpotent, hence solvable.
- This contradicts the fact that $\mathfrak{g}$ is semisimple.