Root systems and semisimple complex Lie algebras

MAT 552

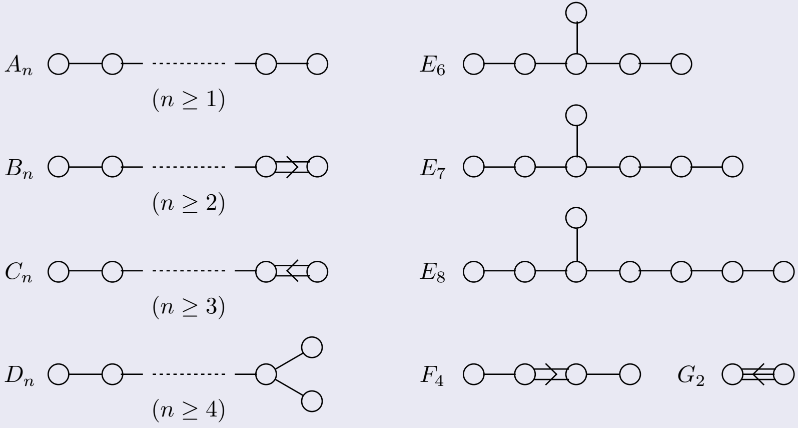
April 29, 2020

Brief review: the classification

Last time, we proved the following theorem.

Theorem

The Dynkin diagram of an irreducible root system is always one of the following diagrams:



Brief review: Dynkin diagrams

Let R be an irreducible root system, with basis $\Pi \subseteq R$.

The Dynkin diagram of R is constructed as follows:

- It has one vertex for every simple root $\alpha \in \Pi$.
- Every pair of vertices $\alpha, \beta \in \Pi$ is connected by

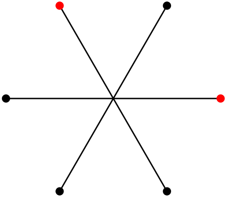
  $$n_{\alpha,\beta} \cdot n_{\beta,\alpha} = 4\cos^2\theta \in \{0, 1, 2, 3\}$$

  distinct edges.
- Each multiple edge has an arrow that points towards the **shorter** root.

We saw last time that the Dynkin diagram determines the root system (up to isomorphism).

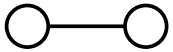
Brief review: examples in rank 2

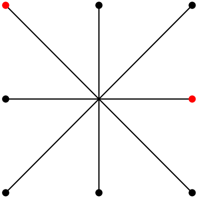
The three irreducible root systems of rank 2:



A_2

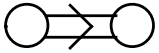
Dynkin diagram:

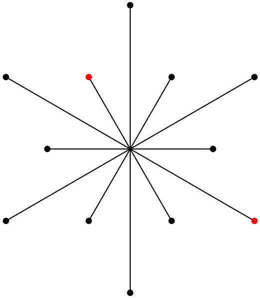




B_2

Dynkin diagram:





G_2

Dynkin diagram:

The classical Lie algebras

The Dynkin diagrams of type A_n, B_n, C_n, and D_n correspond to the classical Lie algebras:

- A_n is the root system of $\mathfrak{sl}(n+1, \mathbb{C})$
- B_n is the root system of $\mathfrak{so}(2n+1, \mathbb{C})$
- C_n is the root system of $\mathfrak{sp}(n, \mathbb{C})$
- D_n is the root system of $\mathfrak{so}(2n, \mathbb{C})$

Some of the diagrams for small n are the same; this reflects "accidental" isomorphisms among some of the Lie algebras.

Example

The Dynkin diagrams of type A_3, D_3 are the same. The reason is the isomorphism of Lie algebras (both 15-dimensional)

$$\mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C}).$$

Example: $\mathfrak{sl}(4,\mathbb{C}) \cong \mathfrak{so}(6,\mathbb{C})$

Recall the definitions of the two Lie algebras:

- $\mathfrak{sl}(4,\mathbb{C})$ are 4×4-matrices x with $\operatorname{tr} x = 0$.
- $\mathfrak{so}(6,\mathbb{C})$ are 6×6-matrices y with $y + y^t = 0$.

Both are clearly 15-dimensional.

The isomorphism comes from the 6-dimensional $V = \bigwedge^2 \mathbb{C}^4$.

- The isomorphism $\det\colon \bigwedge^4 \mathbb{C}^4 \to \mathbb{C}$ defines a pairing
$$V \otimes V \to \mathbb{C}, \quad v \otimes w \mapsto \det(v \wedge w).$$
- This is symmetric and nondegenerate.
- The action by $\mathfrak{sl}(4,\mathbb{C})$ defines a morphism of Lie algebras
$$\mathfrak{sl}(4,\mathbb{C}) \to \mathfrak{so}(6,\mathbb{C}).$$
- This is injective, hence an isomorphism by dimension.

Example: $\mathfrak{so}(2n, \mathbb{C})$

Let us compute the root system in the example $\mathfrak{so}(2n, \mathbb{C})$.
This is the Lie algebra of $2n \times 2n$-matrices y with $y^t + y = 0$.

The subalgebra of block-diagonal matrices of the form

$$\begin{pmatrix} H_1 & & & \\ & H_2 & & \\ & & \ddots & \\ & & & H_n \end{pmatrix}, \quad H_i = \begin{pmatrix} 0 & h_i \\ -h_i & 0 \end{pmatrix},$$

is a Cartan subalgebra (of dimension n).

Example: $\mathfrak{so}(2n, \mathbb{C})$

An alternative description of $\mathfrak{so}(2n, \mathbb{C})$ is

$$\mathfrak{g} = \left\{ x \in \mathfrak{gl}(2n, \mathbb{C}) \;\middle|\; Kx + x^t K = 0 \right\},$$

where K is the block matrix of size $2n \times 2n$ given by

$$K = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The isomorphism takes $x \in \mathfrak{g}$ to the matrix $Kx \in \mathfrak{so}(2n, \mathbb{C})$ (which is skew-symmetric since $(Kx)^t = x^t K = -Kx$).

In this description, the Cartan subalgebra is

$$\mathfrak{h} = \left\{ \operatorname{diag}(h_1, \ldots, h_n, -h_1, \ldots, -h_n) \;\middle|\; h_1, \ldots, h_n \in \mathbb{C} \right\}.$$

Example: $\mathfrak{so}(2n, \mathbb{C})$

The diagonal entries give us linear functionals

$$e_i \colon \mathfrak{h} \to \mathbb{C}, \quad \mathrm{diag}(h_1, \ldots, h_n, -h_1, \ldots, h_n) \mapsto h_i.$$

In this basis, the Killing form is a multiple of the bilinear form

$$(e_i, e_j) = \delta_{i,j}.$$

Recall the following matrices:

$$E_{i,j} = \begin{array}{cl} \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix} & \leftarrow i\text{-th row} \\ \uparrow & \\ j\text{-th column} & \end{array}$$

Example: $\mathfrak{so}(2n, \mathbb{C})$

The root subspaces are easily computed:

- $\mathfrak{g}_{e_i - e_j} = \mathbb{C}(E_{i,j} - E_{j+n,i+n})$
- $\mathfrak{g}_{e_i + e_j} = \mathbb{C}(E_{i,j+n} - E_{j,i+n})$
- $\mathfrak{g}_{-e_i - e_j} = \mathbb{C}(E_{i+n,j} - E_{j+n,i})$

For instance, if $h = \mathrm{diag}(h_1, \dots, h_n, -h_1, \dots, h_n)$, then

$$\begin{aligned}
[h, E_{i,j} - E_{j+n,i+n}] &= (h_i - h_j)E_{i,j} - (-h_j + h_i)E_{j+n,i+n} \\
&= (h_i - h_j)(E_{i,j} - E_{j+n,i+n}) \\
&= \Big(e_i(h) - e_j(h)\Big) \cdot (E_{i,j} - E_{j+n,i+n}).
\end{aligned}$$

The root system is therefore

$$R = \left\{ \pm e_i \pm e_j \;\middle|\; i \neq j \right\},$$

with signs chosen independently. There are $2n(n-1)$ roots.

Example: $\mathfrak{so}(2n, \mathbb{C})$

A natural choice of polarization is

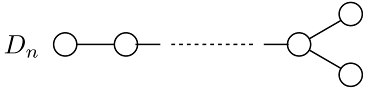
$$R_+ = \left\{ e_i \pm e_j \;\middle|\; i < j \right\}.$$

The set of simple roots is $\Pi = \{\alpha_1, \ldots, \alpha_n\}$:

$$\begin{aligned}
\alpha_1 &= e_1 - e_2 \\
\alpha_2 &= e_2 - e_3 \\
&\vdots \\
\alpha_{n-1} &= e_{n-1} - e_n \\
\alpha_n &= e_{n-1} + e_n
\end{aligned}$$

Example: $\mathfrak{so}(2n, \mathbb{C})$

This gives us the picture of the Dynkin diagram:



Indeed, for $1 \leq i < j \leq n-1$, we have

$$(\alpha_i, \alpha_j) = (e_i - e_{i+1}, e_j - e_{j+1}) = \begin{cases} -1 & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

The branching at α_{n-2} comes from the fact that

$$(\alpha_{n-2}, \alpha_n) = (e_{n-2} - e_{n-1}, e_{n-1} + e_n) = -1$$
$$(\alpha_{n-1}, \alpha_n) = (e_{n-1} - e_n, e_{n-1} + e_n) = 0.$$

Brief review: root decomposition

Goal: Classify (semi-)simple complex Lie algebras

Recall how we got from Lie algebras to Dynkin diagrams:

- Let $\mathfrak{g}$ be a semisimple complex Lie algebra.
- Choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.
- It determines the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

- The subspaces

$$\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \right\}$$

  are the root root subspaces (with $\mathfrak{g}_0 = \mathfrak{h}$).
- The set $R \subseteq \mathfrak{h}^*$ is the root system of $\mathfrak{g}$.

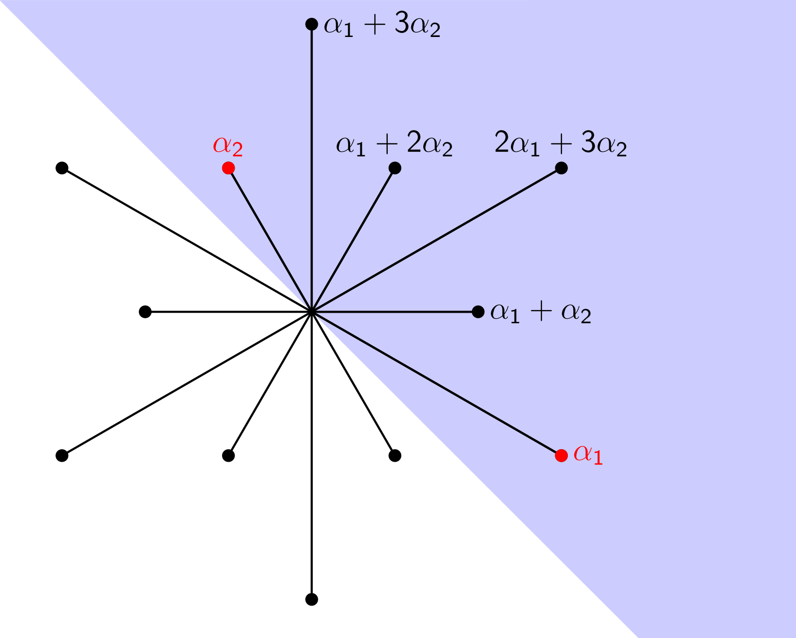
Brief review: root decomposition

We have classified the possible root systems. The question is to what extent the root system R determines the Lie algebra $\mathfrak{g}$.

Recall the following facts about the root subspaces:

1. For each root $\alpha \in R$, one has $\dim \mathfrak{g}_\alpha = 1$.
2. If $\alpha, \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
3. For $\alpha \in R$, there is a distinguished element $h_\alpha \in \mathfrak{h}$ with
$$\beta(h_\alpha) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \quad \text{for all } \beta \in \mathfrak{h}^*.$$
4. For $\alpha \in R$, the subspace
$$\mathfrak{sl}(2, \mathbb{C})_\alpha = \mathfrak{g}_\alpha \oplus \mathbb{C}h_\alpha \oplus \mathfrak{g}_{-\alpha}$$
is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Example: the root system of type G_2



The Serre relations

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Key data:

- a non-degenerate invariant symmetric bilinear form $(-,-)$, for example the Killing form
- a Cartan subalgebra $\mathfrak{h}$
- the root system $R \subseteq \mathfrak{h}^*$
- a polarization $R = R_+ \sqcup R_-$
- the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$

Theorem 1

As a vector space, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and the two subspaces

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$$

are subalgebras of $\mathfrak{g}$.

The Serre relations

Theorem 1

As a vector space, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and the two subspaces

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$$

are subalgebras of $\mathfrak{g}$.

Proof:

- The first part is just the root decomposition.
- We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
- If $\alpha \in R_+$ and $\beta \in R_+$, then $\alpha + \beta \in R_+$.
- Therefore $\mathfrak{n}_+$ is a subalgebra.
- Similarly for $\mathfrak{n}_-$. $\square$

The Serre relations

For each simple root $\alpha_i \in \Pi$, set $h_i = h_{\alpha_i} \in \mathfrak{h}$. Then

$$\mathfrak{g}_{-\alpha_i} \oplus \mathbb{C}h_i \oplus \mathfrak{g}_{\alpha_i}$$

is a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Choosing $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ so that $(e_i, f_i) = 2/(\alpha_i, \alpha_i)$, we get the relations

$$[h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i, \quad [e_i, f_i] = h_i.$$

Theorem 2

As a Lie algebra, $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i\}_{i=1,\ldots,r}$. In fact:

1. The elements $e_1, \ldots, e_n$ generate $\mathfrak{n}_+$.
2. The elements $f_1, \ldots, f_n$ generate $\mathfrak{n}_-$.
3. The elements $h_1, \ldots, h_n$ are a basis of $\mathfrak{h}$.

The Serre relations

Theorem 2

As a Lie algebra, $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i\}_{i=1,\dots,r}$. In fact:

1. The elements $e_1, \dots, e_n$ generate $\mathfrak{n}_+$.
2. The elements $f_1, \dots, f_n$ generate $\mathfrak{n}_-$.
3. The elements $h_1, \dots, h_n$ are a basis of $\mathfrak{h}$.

Proof:

- (3) holds because Π is a basis for $\mathfrak{h}^*$.
- Since $\dim \mathfrak{g}_\alpha = 1$, we have $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$.
- For any two roots $\alpha, \beta \in R$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
- Now (1) follows, because every positive root can be written as a finite sum of simple roots.
- A similar argument proves (2). $\square$

The Serre relations

For two simple roots $\alpha_i, \alpha_j \in \Pi$, we have the integer

$$a_{i,j} = n_{\alpha_j,\alpha_i} = \alpha_j(h_i) \in \mathbb{Z}.$$

We have $a_{i,i} = 2$, and $a_{i,j} \in \{0, -1, -2, -3\}$ if $i \neq j$.

Theorem 3

The elements $\{e_i, f_i, h_i\}_{i=1,\dots,n}$ satisfy the following relations:

1. $[h_i, h_j] = 0$
2. $[h_i, e_j] = a_{i,j} e_j$ and $[h_i, f_j] = -a_{i,j} f_j$
3. $[e_i, f_j] = \delta_{i,j} h_i$
4. $(\mathrm{ad}\, e_i)^{1-a_{i,j}} e_j = 0$ and $(\mathrm{ad}\, f_i)^{1-a_{i,j}} f_j = 0$, for $i \neq j$

The relations in the theorem are known as the Serre relations.

The Serre relations

Theorem 3

The elements $\{e_i, f_i, h_i\}_{i=1,\ldots,n}$ satisfy the following relations:

1. $[h_i, h_j] = 0$
2. $[h_i, e_j] = a_{i,j}e_j$ and $[h_i, f_j] = -a_{i,j}f_j$

Proof:

- (1) is clear: the Cartan subalgebra $\mathfrak{h}$ is commutative.
- Since $e_j \in \mathfrak{g}_{\alpha_j}$, we have

$$[h_i, e_j] = \alpha_j(h_i)e_j = a_{i,j}e_j.$$

- Similarly for $f_j \in \mathfrak{g}_{-\alpha_j}$, and so (2) holds.

The Serre relations

Theorem 3

The elements $\{e_i, f_i, h_i\}_{i=1,\dots,n}$ satisfy the following relations:

3. $[e_i, f_j] = \delta_{i,j} h_i$

Proof:

- (3) holds when $i = j$, because

$$[e_i, f_i] = (e_i, f_i) H_{\alpha_i} = \frac{2}{(\alpha_i, \alpha_i)} H_{\alpha_i} = h_{\alpha_i} = h_i.$$

- When $i \neq j$, we have $[e_i, f_j] \in \mathfrak{g}_{\alpha_i - \alpha_j}$.
- But $\mathfrak{g}_{\alpha_i - \alpha_j} = \{0\}$ because $\alpha_i - \alpha_j \notin R$.
- Otherwise, either $\alpha_i - \alpha_j \in R_+$ or $\alpha_j - \alpha_i \in R_+$, and both contradict α_i and α_j being simple.
- So (3) holds in general.

The Serre relations

Theorem 3

The elements $\{e_i, f_i, h_i\}_{i=1,\dots,n}$ satisfy the following relations:

4. $(\mathrm{ad}\, e_i)^{1-a_{i,j}} e_j = 0$ and $(\mathrm{ad}\, f_i)^{1-a_{i,j}} f_j = 0$, for $i \neq j$

Proof:

- To prove (4), consider the subspace

$$\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{-\alpha_j+k\alpha_i} \subseteq \mathfrak{g}$$

  as a representation of $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i} = \langle e_i, f_i, h_i\rangle$.
- By (3), $\mathrm{ad}\, e_i.f_j = 0$, so f_j is a highest-weight vector.
- By (2), $\mathrm{ad}\, h_i.f_j = -a_{i,j} f_j$, so f_j has weight $w = -a_{i,j}$.
- Thus $(\mathrm{ad}\, f_i)^{1-a_{i,j}} f_j = 0$, because $-w-2$ is not a weight in an irreducible representation of highest weight w.
- The other half is proved similarly. □

The Serre relations

In fact, one can prove the following theorem.

Theorem

Let R be an irreducible root system of rank n with polarization $R = R_+ \sqcup R_-$ and simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$.

Let $\mathfrak{g}(R)$ be the complex Lie algebra with $3n$ generators $\{e_i, f_i, h_i\}_{i=1,\ldots,n}$, subject to the Serre relations:

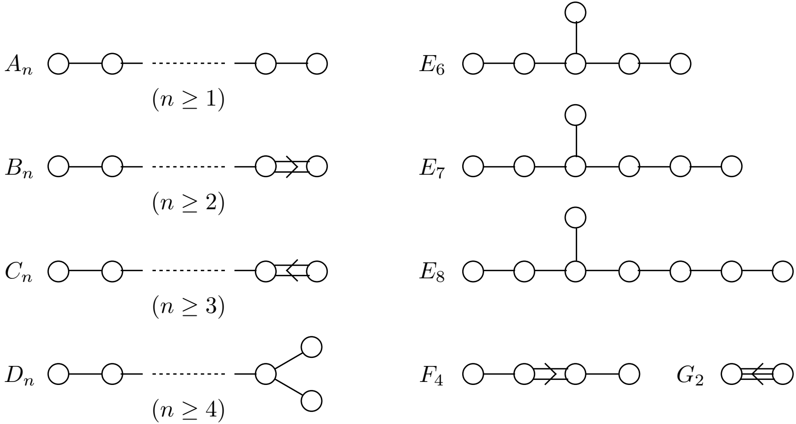
1. $[h_i, h_j] = 0$
2. $[h_i, e_j] = a_{i,j}e_j$ *and* $[h_i, f_j] = -a_{i,j}f_j$
3. $[e_i, f_j] = \delta_{i,j}h_i$
4. $(\mathrm{ad}\, e_i)^{1-a_{i,j}}e_j = 0$ *and* $(\mathrm{ad}\, f_i)^{1-a_{i,j}}f_j = 0$*, for* $i \neq j$

Then $\mathfrak{g}(R)$ is a finite-dimensional simple Lie algebra whose root system is the given root system R.

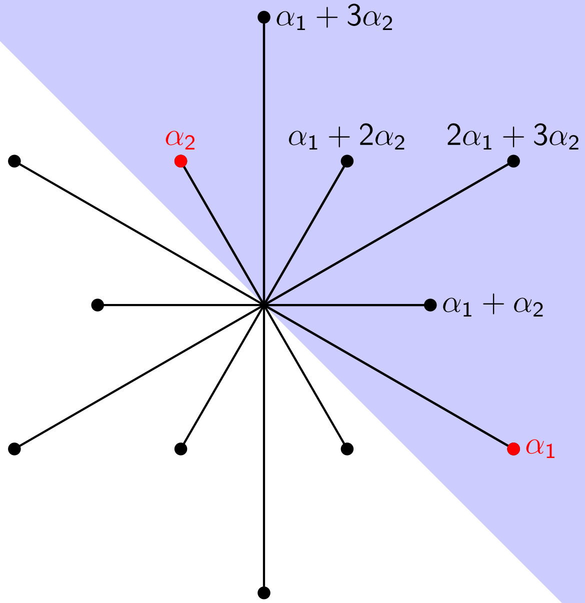
The Serre relations

Corollary

Simple finite-dimensional complex Lie algebras are classified by the Dynkin diagrams A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2.



Example: the root system of type G_2



Example: the root system of type G_2

Let us consider the example of G_2. The integers $a_{i,j}$ are:

$$a_{1,1} = a_{2,2} = 2, \quad a_{1,2} = -1, \quad a_{2,1} = -3.$$

The corresponding Lie algebra has dimension 14. In

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

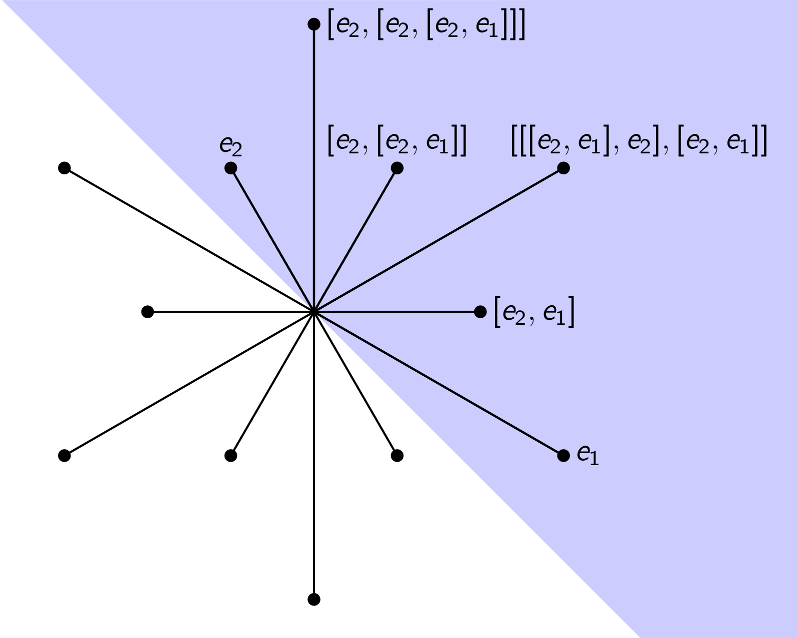
we have $\dim \mathfrak{h} = 2$ and $\dim \mathfrak{n}_\pm = 6$.

The theorem tells us that $\mathfrak{n}_+$ is generated (as a Lie algebra) by two elements e_1, e_2, subject to the two Serre relations

$$(\operatorname{ad} e_1)^2 e_2 = 0 \quad \text{and} \quad (\operatorname{ad} e_2)^4 e_1 = 0.$$

(The picture for $\mathfrak{n}_-$ is similar.)

Example: the root system of type G_2



Example: the root system of type G_2

The picture of the root system shows how

$$\mathfrak{sl}(2,\mathbb{C})_{\alpha_1} = \langle e_1, f_1, h_1 \rangle$$
$$\mathfrak{sl}(2,\mathbb{C})_{\alpha_2} = \langle e_2, f_2, h_2 \rangle$$

act on the Lie algebra $\mathfrak{g}$. The irreducible subrepresentations correspond to strings of roots.

The weight of the root subspace $\mathfrak{g}_\alpha$ with respect to $\operatorname{ad} h_2$ is

$$\alpha(h_2) = (c_1\alpha_1 + c_2\alpha_2)(h_2) = c_1\alpha_1(h_2) + c_2\alpha_2(h_2) = -3c_1 + 2c_2,$$

writing $\alpha = c_1\alpha_1 + c_2\alpha_2$.

Example: the root system of type G_2