The classification theorem

MAT 552

April 27, 2020

Brief review: abstract root systems

E is a finite dimensional real vector space.
$(-,-)$ is a positive definite inner product on E.

A (reduced) root system is a subset $R \subseteq E \setminus \{0\}$ such that:

1. R is finite and spans E.
2. If $\alpha \in R$ and $c\alpha \in R$, then $c = \pm 1$.
3. For every $\alpha, \beta \in R$, one has
$$n_{\beta,\alpha} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}.$$
4. If $\alpha, \beta \in R$, then also $\beta - n_{\beta,\alpha}\,\alpha \in R$.

The rank of the root system is $\dim E$. The Weyl group W is the group generated by the reflections s_α, $\alpha \in R$.

Brief review: simple roots

A choice of polarization divides the root system

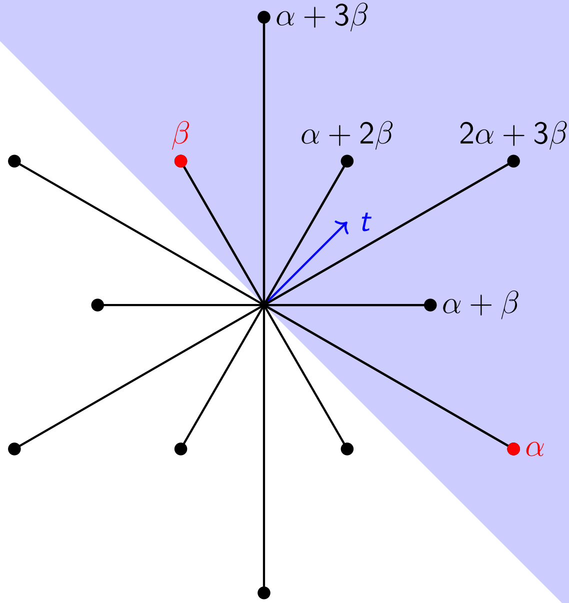
$$R = R_+ \sqcup R_-$$

into positive and negative roots.

A positive root is simple if it is not the sum of two other positive roots. The set of simple roots is denoted Π.

- Π is a basis of the vector space E.
- Any two bases are related by an element of W.
- For simple roots $\alpha, \beta \in \Pi$, one has $(\alpha, \beta) \leq 0$.

Example: simple roots in G_2



Reconstructing R from Π

Last time, we showed that the entire root system R can be reconstructed from the set of simple roots Π:

- The Weyl group W is generated by simple reflections, which are the reflections s_α with $\alpha \in \Pi$.
- One has $R = W(\Pi)$.

Recall that R is called reducible if

$$R = R' \sqcup R'' \quad \text{with } R' \perp R''.$$

This is equivalent to having

$$\Pi = \Pi' \sqcup \Pi'' \quad \text{with } \Pi' \perp \Pi''.$$

If there is no such decomposition, then R is irreducible.

Coxeter graphs and Dynkin diagrams

Goal: Classify irreducible root systems.

Let $\alpha, \beta \in \Pi$ be two simple roots with $\|\alpha\| \geq \|\beta\|$. We know that $(\alpha, \beta) \leq 0$, and of course $\beta \neq -\alpha$. The relation

$$n_{\alpha,\beta} \cdot n_{\beta,\alpha} = 4\cos^2\theta,$$

leaves only four possible values for the angle θ:

$4\cos^2\theta$	$n_{\beta,\alpha}$	$n_{\alpha,\beta}$	$\|\alpha\|/\|\beta\|$	θ
0	0	0	any	$\pi/2$
1	-1	-1	1	$2\pi/3$
2	-1	-2	$\sqrt{2}$	$3\pi/4$
3	-1	-3	$\sqrt{3}$	$5\pi/6$

Everything is determined by the integer $-n_{\alpha,\beta} \in \{0, 1, 2, 3\}$, together with the fact that $\|\alpha\| \geq \|\beta\|$.

Coxeter graphs and Dynkin diagrams

Let R be an irreducible root system.

We can encode the information about the simple roots into a graph; that way, it is easier to see what is going on.

The Coxeter graph of R is actually a "multigraph":

- It has one vertex for every simple root of R.
- Every pair of vertices $\alpha, \beta \in \Pi$ is connected by

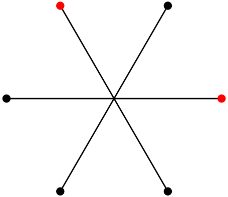
$$n_{\alpha,\beta} \cdot n_{\beta,\alpha} = 4\cos^2\theta \in \{0, 1, 2, 3\}$$

  distinct edges.

The Dynkin diagram of R is the Coxeter graph, but with arrows pointing towards the **shorter** root attached to the double and triple edges.

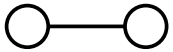
Examples in rank 2

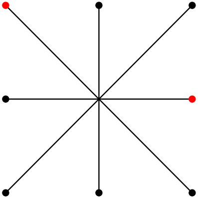
Here are the three irreducible root systems of rank 2:



A_2

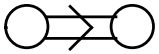
Dynkin diagram:

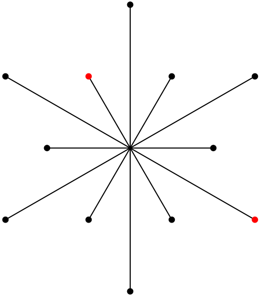




B_2

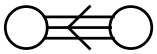
Dynkin diagram:





G_2

Dynkin diagram:



Dynkin diagrams vs. root systems

The Dynkin diagram only depends on the root system R, but not on the choice of basis $\Pi \subseteq R$.

- Any two bases are related by an element of W. $\square$

The Dynkin of an irreducible root system is connected.

- Suppose the Dynkin diagram is not connected.
- Then $\Pi = \Pi' \sqcup \Pi''$, in such a way that there are no edges between vertices in Π' and Π''.
- By definition of the Dynkin diagram, this means that $(\alpha, \beta) = 0$ for every $\alpha \in \Pi'$ and every $\beta \in \Pi''$.
- This contradicts the fact that R is irreducible. $\square$

Dynkin diagrams vs. root systems

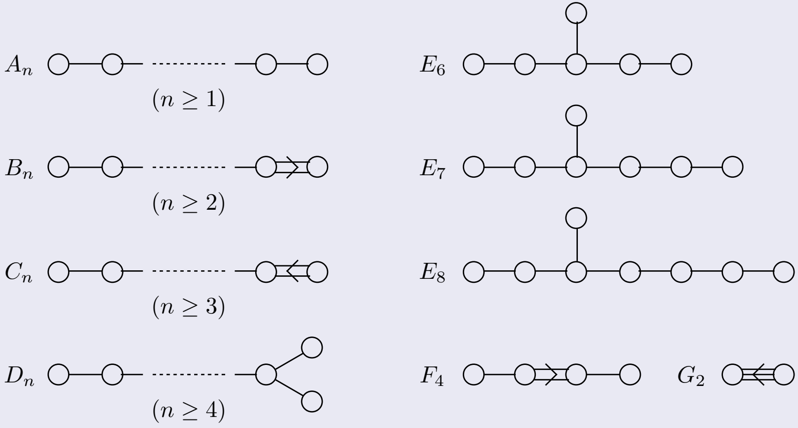
One can reconstruct an irreducible root system from its Dynkin diagram (up to isomorphism).

- Since R is irreducible, the Dynkin diagram is connected.
- So any $\alpha, \beta \in \Pi$ are joined by a sequence of edges.
- Hence the Dynkin diagram determines the inner products (α, β) for all $\alpha, \beta \in \Pi$, up to a common scale factor.
- This information determines E and $(-,-)$.
- The Weyl group W is generated by simple reflections.
- Therefore $R = W(\Pi)$ is determined (up to rescaling). $\square$

The classification theorem

Theorem

The Dynkin diagram of an irreducible root system is always one of the following diagrams:



There are four infinite families and five exceptional types.

Admissible diagrams

It is enough to classify the Coxeter graphs; we can then get the Dynkin diagrams by adding arrows in the right places.

This means that we can forget about the lengths of vectors for now. The following piece of terminology is useful:

Definition

We say that a linearly independent set of unit vectors $v_1, \ldots, v_n \in E$ is an admissible configuration if

1. $(v_i, v_j) \leq 0$ for every $i \neq j$,
2. $4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$ for every $i \neq j$.

The resulting Coxeter graph is called an admissible diagram.

Any subset of an admissible configuration is again admissible. Any subdiagram of an admissible diagram is itself admissible.

Proof of the classification theorem

Step 1: A connected admissible diagram is a tree.

- Let $v_1, \ldots, v_n$ be an admissible configuration.
- Define $v = v_1 + \cdots + v_n$; clearly $v \neq 0$.
- Therefore $(v, v) = \sum_{i,j} (v_i, v_i) = n + \sum_{i<j} 2(v_i, v_j) > 0$.
- If v_i and v_j are connected by an edge, we must have $2(v_i, v_j) \in \{-1, -\sqrt{2}, -\sqrt{3}\}$, hence $-2(v_i, v_j) \geq 1$.
- The diagram can have at most $n-1$ edges, because

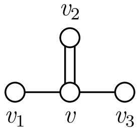
$$n > \sum_{i<j} -2(v_i, v_j) \geq (\text{number of edges}).$$

- Being connected, the diagram must have exactly $n-1$ edges (ignoring multiplicity), hence must be a tree. $\square$

Proof of the classification theorem

Step 2: Each vertex meets ≤ 3 edges (with multiplicity).

- Let v be a vertex.
- Let $v_1, \ldots, v_d$ be the vertices connected to v.



- Since the Dynkin diagram has no cycles (by Step 1), there can be no edge between v_i and v_j for $i \neq j$.
- Therefore $(v_i, v_j) = 0$ for $i \neq j$.
- This says that $v_1, \ldots, v_d$ are orthonormal.
- Since v is **not** in the span of $v_1, \ldots, v_d$, we have

$$v \neq \sum_{i=1}^{d} (v, v_i) v_i.$$

Proof of the classification theorem

Step 2: Each vertex meets ≤ 3 edges (with multiplicity).

- Since v is **not** in the span of $v_1, \ldots, v_d$, we have

$$v \neq \sum_{i=1}^{d} (v, v_i) v_i.$$

- Because v is a unit vector, we get

$$4 = 4(v, v) > \sum_{i=1}^{d} 4(v, v_i)^2.$$

- But $4(v, v_i)^2$ is the number of edges connecting v and v_i.
- So the number of edges starting at v is less than 4. $\square$

Proof of the classification theorem

Step 3: The only connected admissible diagram with a triple edge is the exceptional diagram of type G_2:

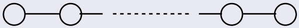
- This is clear from Step 2. $\square$

Note

From now on, we can restrict our attention to admissible diagrams with single and double edges.

Proof of the classification theorem

Step 4: A simple chain $v_1, \ldots, v_k$ in an admissible diagram can always be replaced by the single vertex $v = v_1 + \cdots + v_k$.



The first claim is that v is a unit vector:

- We have $(v, v) = k + \sum_{i<j} 2(v_i, v_j)$.
- Since there are no cycles (by Step 1), we have must $(v_i, v_j) = 0$ for $i < j$, except when $j = i + 1$.
- Consecutive vertices are connected by a single edge, and therefore $2(v_i, v_{i+1}) = -1$ for $i = 1, \ldots, k - 1$.
- This gives $(v, v) = k - (k - 1) = 1$.

Proof of the classification theorem

Step 4: A simple chain $v_1, \ldots, v_k$ in an admissible diagram can always be replaced by the single vertex $v = v_1 + \cdots + v_k$.

The second claim is that the new diagram is still admissible:

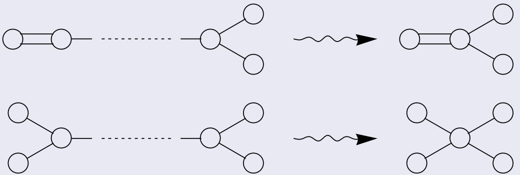
- Since there are no cycles, any vertex u not in the chain is connected to at most one vertex in the chain.
- If $(u, v_j) \neq 0$, say, then

$$(u, v) = \sum_{i=1}^{k} (u, v_i) = (u, v_j).$$

- In the new diagram, u therefore connects to v in the same way as it originally connected to v_j.
- Thus the new diagram is still admissible. $\square$

Proof of the classification theorem

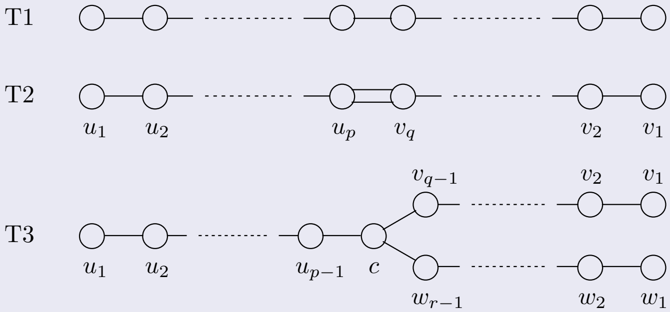
Step 5: A connected admissible diagram cannot contain any of the following subdiagrams:



- By Step 4, such a subdiagram can be collapsed.
- This makes a vertex of degree 4, contradicting Step 2. $\square$

Proof of the classification theorem

Step 6: There are only three possible types of connected admissible diagrams (without triple edges):

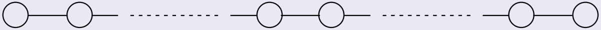


- By Step 5, there can be at most one double edge and at most one branching, but not both at the same time. $\square$

Proof of the classification theorem

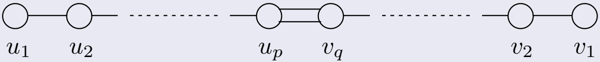
Step 6: The admissible diagram of type T1 corresponds to the Dynkin diagram A_n in the theorem (with $n \geq 1$).

T1



Step 7: The only admissible diagrams of type T2 are B_n, C_n (equal except for the direction of the arrow), and F_4.

T2



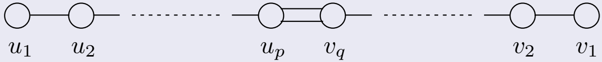
Concretely, this means that there are only two options:

- p arbitrary and $q = 1$ (or vice versa)
- $p = q = 2$

Proof of the classification theorem

Step 7: The only admissible diagrams of type T2 are B_n, C_n (equal except for the direction of the arrow), and F_4.

T2



- Define $u = \sum_{i=1}^{p} i \cdot u_i$ and $v = \sum_{j=1}^{q} j \cdot v_j$.
- A short calculation shows that

$$(u, u) = \frac{p(p+1)}{2} \quad \text{and} \quad (v, v) = \frac{q(q+1)}{2}.$$

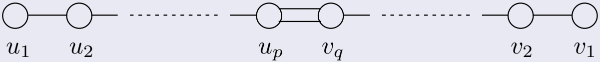
- The double edge is the only edge joining u_i and v_j, so

$$(u, v)^2 = p^2 q^2 (u_p, v_q)^2 = \frac{p^2 q^2}{2}.$$

Proof of the classification theorem

Step 7: The only admissible diagrams of type T2 are B_n, C_n (equal except for the direction of the arrow), and F_4.

T2



- The Cauchy-Schwarz inequality reads

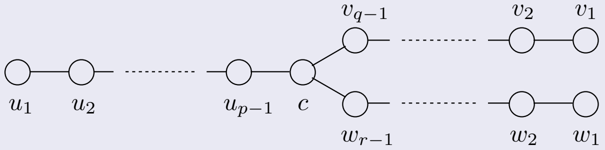
$$(u, v)^2 < (u, u)(v, v),$$

  because u and v are clearly linearly independent.
- Therefore $\dfrac{p^2q^2}{2} < \dfrac{p(p+1)}{2}\,\dfrac{q(q+1)}{2}$.
- This simplifies to $(p-1)(q-1) < 2$.
- So either $p = 1$, or $q = 1$, or $p = q = 2$. $\square$

Proof of the classification theorem

Step 8: The only admissible diagrams of type T3 are D_n (with $n \geq 4$), and E_6, E_7, and E_8.

T3



- By a similar method as in Step 7, one shows that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

- Without loss of generality $p \geq q \geq r \geq 2$.
- The inequality forces $r = 2$, and $q \in \{2, 3\}$.
- If $q = 2$, then $p \geq 2$ is arbitrary $\Longrightarrow D_n$.
- If $q = 3$, then $p \in \{3, 4, 5\} \Longrightarrow E_6, E_7, E_8$. $\square$