# Simple roots and the Weyl group

MAT 552

April 22, 2020

# Brief review: abstract root systems

$E$ is a finite dimensional real vector space.
$(-,-)$ is a positive definite inner product on $E$.

A (reduced) root system is a subset $R \subseteq E \setminus \{0\}$ such that:

1. $R$ is finite and spans $E$.
2. If $\alpha \in R$ and $c\alpha \in R$, then $c = \pm 1$.
3. For every $\alpha, \beta \in R$, one has
$$n_{\beta,\alpha} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$
4. If $\alpha, \beta \in R$, then also $\beta - n_{\beta,\alpha}\,\alpha \in R$.

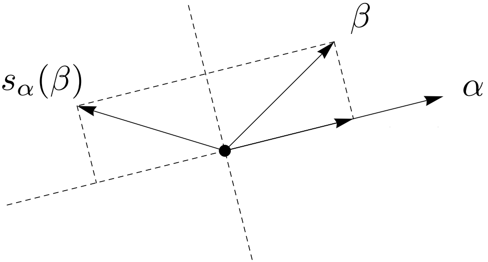
The rank of the root system is $\dim E$.

# Brief review: reflections

Recall the geometric meaning of (4). Let

$$s_\alpha \colon E \to E, \quad s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha,$$

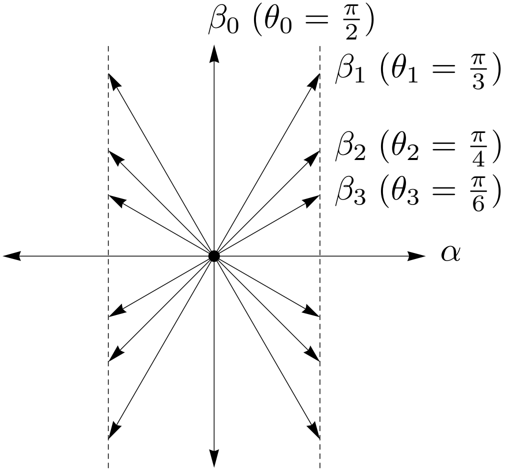
be the reflection in the hyperplane orthogonal to $\alpha \in R$.




Then (4) is saying that $R$ is closed under $s_\alpha$.

# Brief review: angles and relative lengths

The condition $n_{\beta,\alpha} = 2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$ constrains the angle between roots, as well as the relative lengths of roots.



# Brief review: Weyl group

**Definition**

The Weyl group of a root system $R \subseteq E$ is the subgroup $W \subseteq \mathrm{GL}(E)$ generated by all the reflections $s_\alpha$, for $\alpha \in R$.

The Weyl group is the symmetry group of the root system – not literally, but morally.

**Lemma**

1. *The Weyl group is a finite subgroup of the orthogonal group $O(E)$, and $R$ is invariant under the action of $W$.*
2. *For any $w \in W$ and any $\alpha \in R$, we have $s_{w(\alpha)} = w s_\alpha w^{-1}$.*

# Polarizations

**Goal:** Find a "basis" for the root system $R$.

Choose a vector $t \in E$ such that $(\alpha, t) \neq 0$ for every $\alpha \in R$. The choice of $t$ defines a decomposition

$$R = R_+ \sqcup R_-$$

of the root system into positive and negative roots:

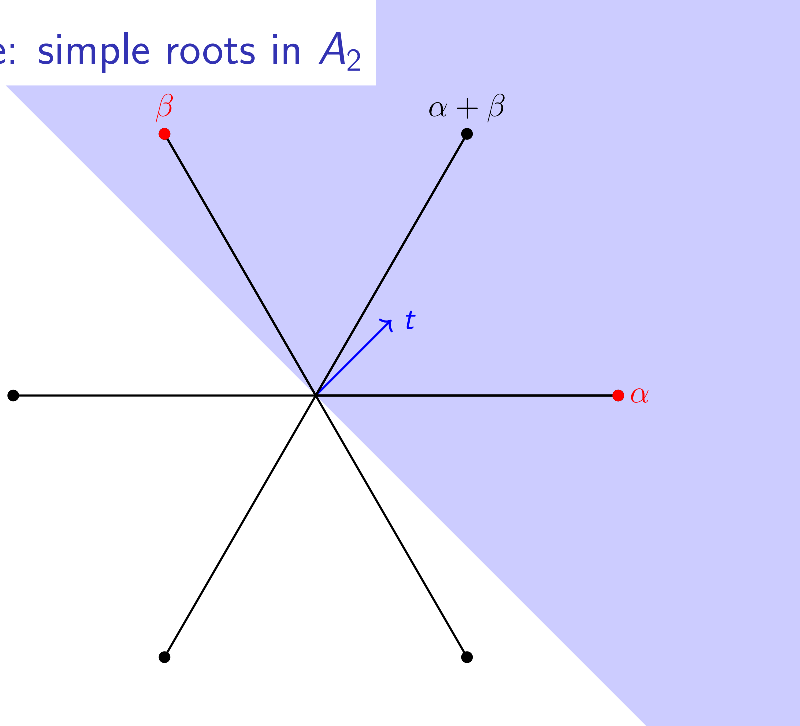
$$R_+ = \left\{ \alpha \in R \;\middle|\; (\alpha, t) > 0 \right\}$$
$$R_- = \left\{ \alpha \in R \;\middle|\; (\alpha, t) < 0 \right\}$$

This decomposition is called a polarization.

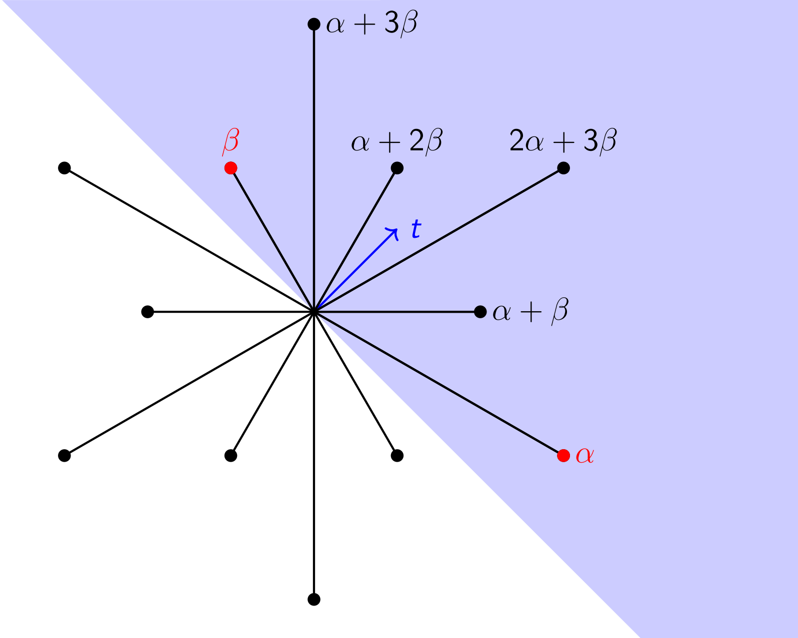
**Definition**

A positive root $\alpha \in R_+$ is called simple if it is not the sum of two other positive roots. The set of simple roots is denoted $\Pi$.

# Example: simple roots in $A_2$



# Example: simple roots in $G_2$



# Simple roots

**Theorem**

*The simple roots $\Pi$ form a basis of the vector space $E$.*

We first prove two small lemmas.

**Lemma 1**

If $\alpha, \beta \in R$ and $(\alpha, \beta) > 0$, then $\alpha - \beta \in R$ and $\beta - \alpha \in R$.

- Since $(\alpha, \beta) > 0$, we get $n_{\beta,\alpha} \geq 1$ and $n_{\alpha,\beta} \geq 1$.
- But $n_{\alpha,\beta} n_{\beta,\alpha} = 4\cos^2\theta$, and so $n_{\alpha,\beta} = 1$ or $n_{\beta,\alpha} = 1$.
- If $n_{\alpha,\beta} = 1$, then $s_\beta(\alpha) = \alpha - n_{\alpha,\beta}\beta = \alpha - \beta \in R$.
- If $n_{\beta,\alpha} = 1$, then $s_\alpha(\beta) = \beta - \alpha \in R$.
- The negative of every root is again a root. □

# Simple roots

**Theorem**

*The simple roots $\Pi$ form a basis of the vector space $E$.*

We first prove two small lemmas.

**Lemma 2**

If $\alpha \neq \beta$ are distinct simple roots, then $(\alpha, \beta) \leq 0$.

- Suppose to the contrary that $(\alpha, \beta) > 0$.
- By Lemma 1, we have $\alpha - \beta \in R$ and $\beta - \alpha \in R$.
- Therefore either $\alpha - \beta \in R_+$ or $\beta - \alpha \in R_+$.
- If $\alpha - \beta \in R_+$, then $\alpha = (\alpha - \beta) + \beta$ contradicts $\alpha \in \Pi$.
- Same story if $\beta - \alpha \in R_+$. □

# Simple roots

**Theorem**

*The simple roots $\Pi$ form a basis of the vector space $E$.*

**Step 1:** The simple roots span $E$.

- Every positive root is a sum of simple roots (obvious).
- By definition, $R = R_+ \sqcup R_-$ spans $E$, and $R_- = -R_+$.

**Step 2:** The simple roots are linearly independent.

- Let $\alpha_1, \ldots, \alpha_n \in \Pi$ be the simple roots.
- Since $\alpha_i \in R_+$, we have $(\alpha_i, t) > 0$ for all $i$.
- By Lemma 2, we have $(\alpha_i, \alpha_j) \leq 0$ for $i \neq j$.
- By HW #7.3, this implies $\alpha_1, \ldots, \alpha_n$ linearly independent.

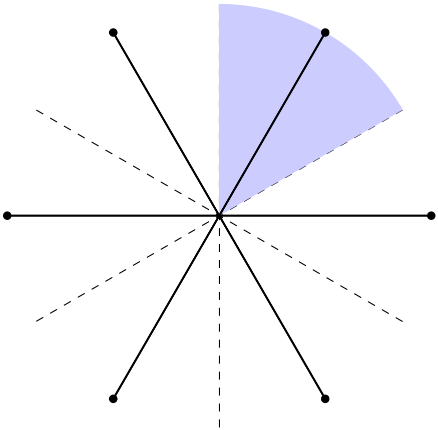
We may say that $\Pi$ is a basis of the root system $R$.

# Weyl chambers

For each root $\alpha \in R$, we have the orthogonal hyperplane

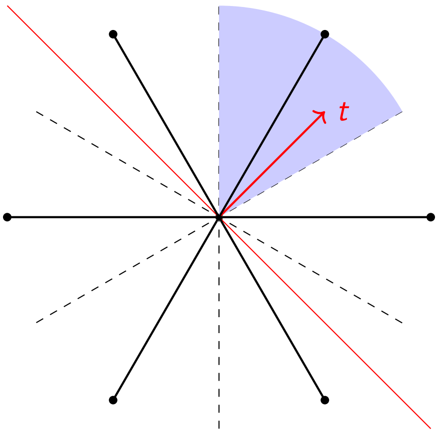
$$L_\alpha = \left\{ x \in E \;\middle|\; (\alpha, x) = 0 \right\}.$$

They divide $E$ into disjoint open cones called Weyl chambers.



# Weyl chambers and bases

How does the set of simple roots depend on $t$?




As long as $t$ stays in one Weyl chamber, the positive roots $R_+$ and hence the simple roots $\Pi$ are unchanged.

# Weyl chambers and bases

We have a bijection between bases and Weyl chambers:

Any Weyl chamber $C \subseteq E$ determines a polarization with

$$R_+ = \left\{ \alpha \in R \;\middle|\; (\alpha, x) > 0 \text{ for all } x \in C \right\},$$

and hence a set of simple roots $\Pi \subseteq R$.

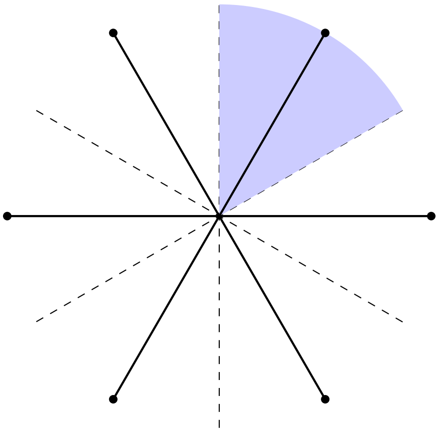
Conversely, a basis $\Pi \subseteq R$ picks out a positive Weyl chamber

$$C_+ = \left\{ x \in E \;\middle|\; (\alpha, x) > 0 \text{ for all } \alpha \in \Pi \right\}.$$

These two constructions are clearly inverse to each other.

# Weyl chambers and bases

It is easy to see that the Weyl group $W$ acts transitively on the set of Weyl chambers (by reflections).



**Conclusion**

Any two bases $\Pi, \Pi' \subseteq R$ are related by an element of $W$.

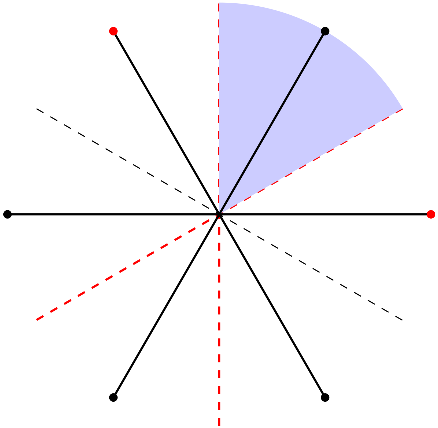
# Simple reflections

**Goal:** Reconstruct $R$ from the set of simple roots $\Pi$.

Let $\alpha_1, \ldots, \alpha_n \in \Pi$ be the simple roots ($n = \dim E$). Let

$$s_i = s_{\alpha_i} \in W$$

be the corresponding simple reflections.



# Simple reflections

**Theorem**

1. *We have* $R = W(\Pi)$.
2. *The simple reflections* $s_1, \ldots, s_n$ *generate* $W$.

This allows us to reconstruct $R$ from $\Pi$:

- The set $\Pi$ determines the simple reflections $s_1, \ldots, s_n$.
- $W \subseteq O(E)$ is the subgroup generated by $s_1, \ldots, s_n$.
- We now recover the entire root system as $R = W(\Pi)$.

**Conclusion (for next time)**

We only need to classify the possible sets of simple roots!

# Example: simple reflections in $A_{n-1}$

Recall that for the root system of $\mathfrak{sl}(n, \mathbb{C})$, we have

$$E = \left\{ x \in \mathbb{R}^n \;\middle|\; x_1 + \cdots + x_n = 0 \right\},$$
$$R = \left\{ e_i - e_j \;\middle|\; i \neq j \right\}.$$

A natural choice of polarization is $R_+ = \left\{ e_i - e_j \;\middle|\; i < j \right\}$.
The resulting set of simple roots is

$$\Pi = \left\{ e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n \right\}.$$

The simple reflections are the transpositions

$$(1, 2) \quad (2, 3) \quad \ldots \quad (n-1, n).$$

These do generate the full symmetric group $W = S_n$.

# Simple reflections

**Theorem**

1. *We have* $R = W(\Pi)$*.*
2. *The simple reflections* $s_1, \dots, s_n$ *generate* $W$*.*

- Let $R' \subseteq R$ be the subset generated by applying any number of simple reflections to the roots in $\Pi$.
- Any $\beta \in R'$ can be written in the form $\beta = w(\alpha_j)$ for some $\alpha_j \in \Pi$ and some $w = s_{i_1} \cdots s_{i_\ell} \in W$.
- By a result from last time,

$$s_\beta = s_{w(\alpha_j)} = w s_j w^{-1} = s_{i_1} \cdots s_{i_\ell} s_j s_{i_\ell} \cdots s_{i_1}$$

  is a product of simple reflections, hence $s_\beta(R') = R'$.
- This proves that $R'$ is again a root system, whose Weyl group $W'$ is the subgroup of $W$ generated by $s_1, \dots, s_n$.

# Simple reflections

**Theorem**

1. *We have* $R = W(\Pi)$.
2. *The simple reflections* $s_1, \ldots, s_n$ *generate* $W$.

- For the induced polarization on $R'$, the set of simple roots is still $\Pi \subseteq R'$.
- Therefore $C_+$ is still one of the Weyl chambers of $R'$.
- Since $W'$ acts transitively on Weyl chambers, it follows that $R$ and $R'$ have the same Weyl chambers.
- Therefore each hyperplane $L_\alpha$ with $\alpha \in R$ must equal $L_\beta$ for some $\beta \in R'$.
- But $L_\alpha$ determines $\pm\alpha$.
- Therefore $R' = R$ and $W' = W$. □

# Weyl group of $G_2$

**Quiz question:** What is the Weyl group of $G_2$?