Abstract root systems

MAT 552

April 20, 2020

Brief review

Last time, we studied the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

in a semisimple complex Lie algebra $\mathfrak{g}$.

Recall that $R \subseteq \mathfrak{h}^*$ is the root system of $\mathfrak{g}$, and

$$\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \right\}$$

are the root root subspaces (with $\mathfrak{g}_0 = \mathfrak{h}$).

Brief review

Concerning the root subspaces

$$\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \right\},$$

we proved the following:

1. For each root $\alpha \in R$, one has $\dim \mathfrak{g}_\alpha = 1$.
2. If $\alpha, \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
3. For $\alpha \in R$, there is a distinguished element $h_\alpha \in \mathfrak{h}$ with

$$\beta(h_\alpha) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \quad \text{for all } \beta \in \mathfrak{h}^*.$$

4. For $\alpha \in R$, the subspace

$$\mathfrak{g}_\alpha \oplus \mathbb{C}h_\alpha \oplus \mathfrak{g}_{-\alpha}$$

is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Brief review

Concerning the root system $R \subseteq \mathfrak{h}^*$, we proved the following:

1. One has $\mathfrak{h}^* = E \oplus iE$, where $E \subseteq \mathfrak{h}^*$ denotes the real subspace spanned by the root system R.
2. The bilinear form $(-,-)$ is positive definite on E.
3. For every $\alpha, \beta \in R$, one has
$$\beta(h_\alpha) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$
4. For $\alpha \in R$, define the reflection operator $s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$ by
$$s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha.$$
If $\beta \in R$, then also $s_\alpha(\beta) \in R$.
5. The only multiples of $\alpha \in R$ that are also roots are $\pm\alpha$.

Abstract root systems

Let E be a finite dimensional real vector space.
Let $(-,-)$ be a positive definite inner product on E.

A subset $R \subseteq E \setminus \{0\}$ is called a (reduced) root system if

1. R is finite and spans E.
2. If $\alpha \in R$ and $c\alpha \in R$, then $c = \pm 1$.
3. For every $\alpha, \beta \in R$, one has

$$n_{\beta,\alpha} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

4. If $\alpha, \beta \in R$, then also $\beta - n_{\beta,\alpha}\,\alpha \in R$.

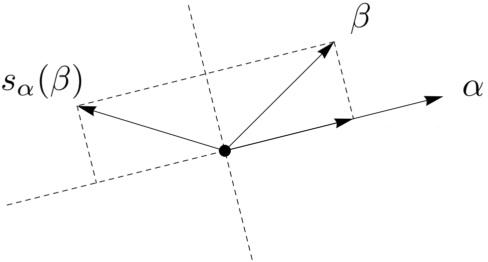
The number $\dim E$ is called the rank of the root system.

Reflections

The geometric meaning of (4) is the following. Let

$$s_\alpha \colon E \to E, \quad s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha,$$

be the reflection in the hyperplane orthogonal to $\alpha \in R$.



Then (4) is saying that R is closed under s_α.

The root system A_{n-1}

Recall our computation of the root system of $\mathfrak{sl}(n, \mathbb{C})$:

$$E = \mathbb{R}^n / \mathbb{R}(1, 1, \ldots, 1) \cong \left\{ x \in \mathbb{R}^n \;\middle|\; x_1 + \cdots + x_n = 0 \right\},$$

with $(-, -)$ induced by the standard inner product on $\mathbb{R}^n$.

The set of roots is

$$R = \left\{ e_i - e_j \;\middle|\; i \neq j \right\}.$$

It consists of $n(n-1)$ vectors, each of length $\sqrt{2}$.

The reflection corresponding to $e_i - e_j$ swaps the i-th and j-th coordinate of a vector.

This root system is called A_{n-1}.

Angles and lengths

The most restrictive condition in the definition is that

$$n_{\beta,\alpha} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

Let $\theta \in [0,\pi]$ be the angle between α and β. Then

$$n_{\beta,\alpha} = \frac{2\|\alpha\|\|\beta\|\cos\theta}{\|\alpha\|^2} = \frac{2\|\beta\|\cos\theta}{\|\alpha\|},$$

and therefore $4\cos^2\theta = n_{\beta,\alpha} n_{\alpha,\beta} \in \mathbb{Z}$.

The only possibilities are $4\cos^2\theta \in \{0,1,2,3,4\}$.

Angles and lengths

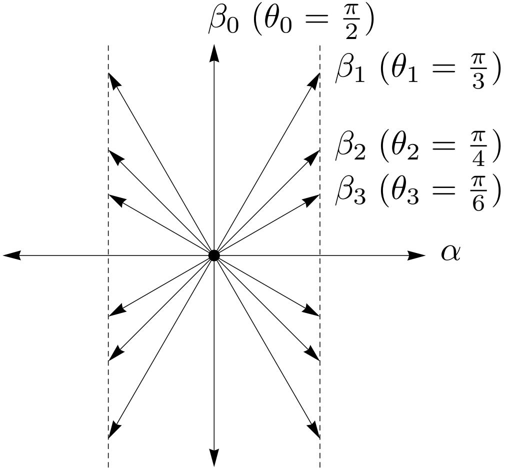
We can list all possibilities, assuming that α is longer than β:

$4\cos^2\theta$	$n_{\beta,\alpha}$	$n_{\alpha,\beta}$	$\|\alpha\|/\|\beta\|$	θ
0	0	0	any	$\pi/2$
1	1	1	1	$\pi/3$
1	-1	-1	1	$2\pi/3$
2	1	2	$\sqrt{2}$	$\pi/4$
2	-1	-2	$\sqrt{2}$	$3\pi/4$
3	1	3	$\sqrt{3}$	$\pi/6$
3	-1	-3	$\sqrt{3}$	$5\pi/6$
4	2	2	1	0
4	-2	-2	1	π

Remember that $\pm\alpha$ are the only possible multiples of α.

Angles and lengths

Here is the same information in pictorial form:



Examples in rank 1

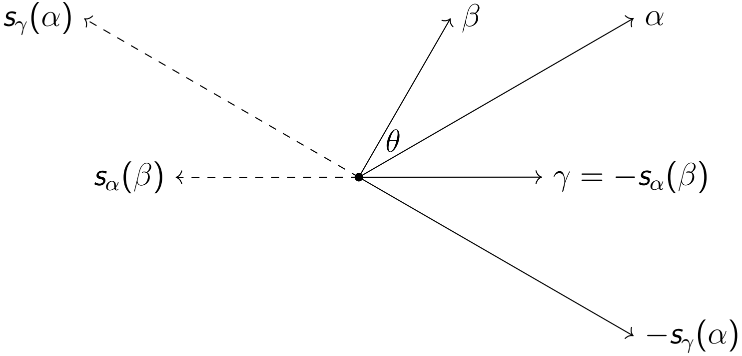
Every root system of rank 1 looks like this:

The only roots are $\pm\alpha$ (but the length of α is arbitrary).

This is the root system A_1, up to rescaling.

Examples in rank 2

In rank 2, the root system is almost completely determined by the angle θ between adjacent roots:

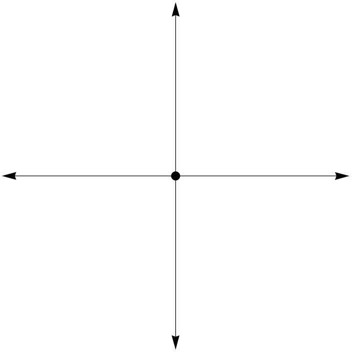


In fact, the angle between any two adjacent roots must be θ.

Examples in rank 2

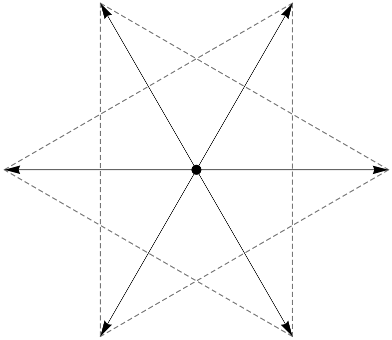
In rank 2, the root system is almost completely determined by the angle θ between adjacent roots.

If $\theta = \pi/2$, we get $A_1 \times A_1$:



The two lengths are arbitrary.

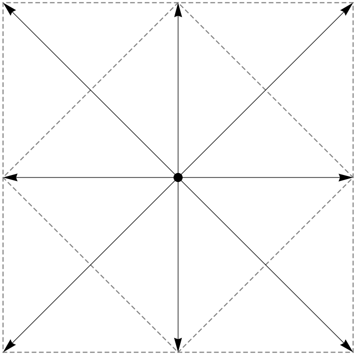
If $\theta = \pi/3$, we get A_2:



Ratio of lengths is 1.

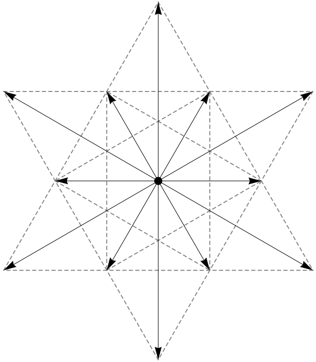
Examples in rank 2

If $\theta = \pi/4$, we get B_2:



Ratio of lengths is $\sqrt{2}$.

If $\theta = \pi/6$, we get G_2:



Ratio of lengths is $\sqrt{3}$.

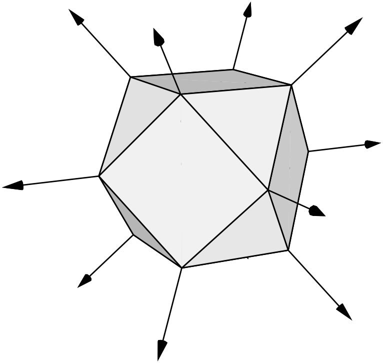
Quiz question

G_2 is the root system of a Lie algebra $\mathfrak{g}$. How big is $\dim \mathfrak{g}$?

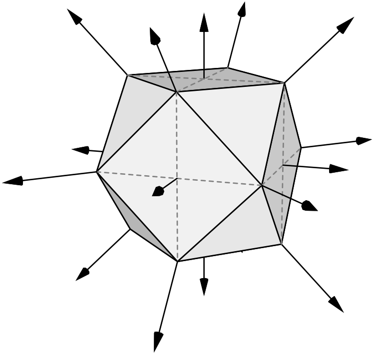
Examples in rank 3

In rank 3, there are three new irreducible examples (which are not products of root systems of lower rank).

The root system A_3:



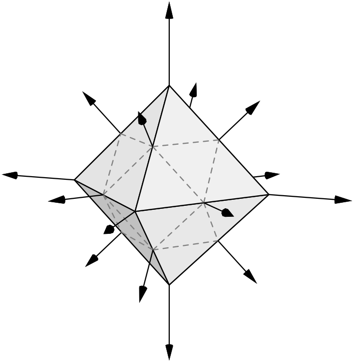
The root system B_3:



Examples in rank 3

In rank 3, there are three new irreducible examples (which are not products of root systems of lower rank).

The root system C_3:



Isomorphisms

We are interested in the numbers $n_{\beta,\alpha}$, more than in the lengths of individual roots.

Definition

Two root systems $R_1 \subseteq E_1$ and $R_2 \subseteq E_2$ are isomorphic if there is a vector space isomorphism

$$\varphi \colon E_1 \to E_2$$

such that $\varphi(R_1) = R_2$ and $n_{\varphi(\beta),\varphi(\alpha)} = n_{\beta,\alpha}$ for all $\alpha, \beta \in R_1$.

Isomorphisms do not need to preserve the inner product.

Example

The root systems R and cR (with $c > 0$) are isomorphic.

Reducible and irreducible root systems

Example

If $R_1 \subseteq E_1$ and $R_2 \subseteq E_2$ are root systems, then

$$R_1 \times \{0\} \cup \{0\} \times R_2 \subseteq E_1 \oplus E_2$$

is another root system, denoted $R_1 \times R_2$. (The inner product on $E_1 \oplus E_2$ is the one where E_1 and E_2 are perpendicular.)

A root system R is reducible if it can be written as a product of two smaller root systems in a nontrivial way. Equivalently,

$$R = R_1 \sqcup R_2$$

with $R_1 \perp R_2$. If this is not possible, R is called irreducible.

Weyl group

Definition

The Weyl group of a root system $R \subseteq E$ is the subgroup $W \subseteq \mathrm{GL}(E)$ generated by all the reflections s_α, for $\alpha \in R$.

One can think of they Weyl group as being a sort of "automorphism group" of the root system.

Example

Let R be the root system of type A_{n-1}.

- The reflection corresponding to $e_i - e_j$ swaps the i-th and j-th coordinate of each vector.
- In coordinates, it is the transposition $(i\, j)$.
- The symmetric group is generated by transpositions.
- Therefore $W \cong S_n$ in this case.

Weyl group

Lemma

1. *The Weyl group is a finite subgroup of the orthogonal group $O(E)$, and R is invariant under the action of W.*
2. *For any $w \in W$ and any $\alpha \in R$, we have $s_{w(\alpha)} = w s_\alpha w^{-1}$.*

- Every reflection s_α is an orthogonal transformation.
- Therefore $W \subseteq O(E)$.
- By the axioms, $s_\alpha(R) = R$, hence $w(R) = R$ for $w \in W$.
- If $w \in W$ leaves every $\alpha \in R$ invariant, then $w = \mathrm{id}$ (because R spans the vector space E).
- Since R is a finite set, W must be a finite group.

Weyl group

Lemma

1. *The Weyl group is a finite subgroup of the orthogonal group $O(E)$, and R is invariant under the action of W.*
2. *For any $w \in W$ and any $\alpha \in R$, we have $s_{w(\alpha)} = ws_\alpha w^{-1}$.*

- For $w \in W$ and $\alpha \in R$, consider $\varphi = ws_\alpha w^{-1}$.
- Let L_α be the hyperplane orthogonal to α.
- Since $w \in O(E)$, we have $w(L_\alpha) = L_{w(\alpha)}$.
- Clearly φ acts as the identity on $L_{w(\alpha)}$.
- Also $\varphi(w(\alpha)) = w(s_\alpha(\alpha)) = -w(\alpha)$.
- Therefore φ is the reflection in the hyperplane $L_{w(\alpha)}$. $\square$

Quiz question

Do isomorphic root sytems have isomorphic Weyl groups?