More about root decompositions

MAT 552

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Brief review

Like last time, $\mathfrak{g}$ is always a semisimple complex Lie algebra.

We fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$:

- $\mathfrak{h}$ is commutative
- every $h \in \mathfrak{h}$ is a semisimple element of $\mathfrak{g}$
- $\mathfrak{h}$ is maximal with these properties

This gives us the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Here $R \subseteq \mathfrak{h}^*$ is the root system of $\mathfrak{g}$, and

$$\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \right\}$$

are the root subspaces. Recall that $\mathfrak{g}_0 = C(\mathfrak{h}) = \mathfrak{h}$.

Brief review

Fix a nondegenerate symmetric invariant bilinear form $(-,-)$ on $\mathfrak{g}$, for example the Killing form K.

Last time, we showed that the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

has the following properties:

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
2. If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under $(-,-)$.
3. $(-,-)$ induces nondegenerate pairings $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$.
4. The restriction of $(-,-)$ to $\mathfrak{h}$ is nondegenerate.

The induced pairing on $\mathfrak{h}^*$

Today, we are going to study the root decomposition in more detail. Our main tool is the representation theory of $\mathfrak{sl}(2, \mathbb{C})$.

First, some notation. We know that $(-,-)$ restricts to a nondegenerate pairing on $\mathfrak{h}$. This gives us an isomorphism

$$\mathfrak{h} \to \mathfrak{h}^*, \quad h \mapsto (h, -).$$

For a linear functional $\alpha \in \mathfrak{h}^*$, we denote by $H_\alpha \in \mathfrak{h}$ the corresponding element of the Cartan subalgebra. Thus

$$(H_\alpha, h) = \alpha(h) \quad \text{for all } h \in \mathfrak{h}.$$

We also get an induced pairing $(-,-)$ on $\mathfrak{h}^*$; concretely,

$$(\alpha, \beta) = (H_\alpha, H_\beta) = \alpha(H_\beta) = \beta(H_\alpha).$$

Finding representations of $\mathfrak{sl}(2,\mathbb{C})$

Lemma

Let $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$ be arbitrary. Then $[e,f] = (e,f)H_\alpha$.

Recall that $H_\alpha \in \mathfrak{h}$ is the unique element with

$$(H_\alpha, h) = \alpha(h) \quad \text{for all } h \in \mathfrak{h}.$$

Since $(-,-)$ is invariant, we have for arbitrary $h \in \mathfrak{h}$ that

$$([e,f],h) = -(f,[e,h]) = (f,[h,e]).$$

Now $e \in \mathfrak{g}_\alpha$ means that $[h,e] = \alpha(h)e = (H_\alpha,h)e$. Thus

$$([e,f],h) = (H_\alpha,h)(f,e) = \big((e,f)H_\alpha, h\big).$$

Since $(-,-)$ is nondegenerate, we get the result. $\square$

Finding representations of $\mathfrak{sl}(2, \mathbb{C})$

Recall that $H_\alpha \in \mathfrak{h}$ is the unique element with

$$(H_\alpha, h) = \alpha(h) \quad \text{for all } h \in \mathfrak{h}.$$

The induced pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ is nondegenerate, so we can choose $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$ with $(e, f) \neq 0$. Then

$$\begin{aligned} [e, f] &= (e, f)H_\alpha \\ [H_\alpha, e] &= \alpha(H_\alpha)e = (\alpha, \alpha)e \\ [H_\alpha, f] &= -\alpha(H_\alpha)f = -(\alpha, \alpha)f \end{aligned}$$

This almost looks like the relations in $\mathfrak{sl}(2, \mathbb{C})$...

We could rescale them if we knew that $(\alpha, \alpha) \neq 0$.

Finding representations of $\mathfrak{sl}(2, \mathbb{C})$

Lemma

For any root $\alpha \in R$, we have $(\alpha, \alpha) = (H_\alpha, H_\alpha) \neq 0$.

- Suppose to the contrary that $(\alpha, \alpha) = 0$.
- Then $\alpha(H_\alpha) = (H_\alpha, H_\alpha) = (\alpha, \alpha) = 0$.
- Choose $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$ with $(e, f) \neq 0$.
- Set $h = [e, f] = (e, f)H_\alpha$ (by the lemma).
- We have $[h, e] = \alpha(h)e = 0$ and $[h, f] = -\alpha(h)f = 0$.
- Therefore $\langle e, f, h \rangle \subseteq \mathfrak{g}$ is a solvable Lie subalgebra.
- By Lie's theorem, $\operatorname{ad} e$, $\operatorname{ad} f$, $\operatorname{ad} h$ are upper triangular in a suitable basis of $\mathfrak{g}$.
- This makes $\operatorname{ad} h = [\operatorname{ad} e, \operatorname{ad} f]$ nilpotent.
- But $\operatorname{ad} h$ is semisimple (because $h \in \mathfrak{h}$), and so $h = 0$.
- This contradicts $(e, f) \neq 0$. $\square$

Finding representations of $\mathfrak{sl}(2,\mathbb{C})$

Let $\alpha \in R$ be any root. Since $(\alpha,\alpha) \neq 0$, we can define

$$h_\alpha = \frac{2}{(\alpha,\alpha)} H_\alpha \in \mathfrak{h}.$$

Note that $\alpha(h_\alpha) = 2$, because $\alpha(H_\alpha) = (\alpha,\alpha)$.
Choose $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $(e_\alpha, f_\alpha)(\alpha,\alpha) = 2$.
Then

$$\begin{aligned} [e_\alpha, f_\alpha] &= (e_\alpha, f_\alpha) H_\alpha = (e_\alpha, f_\alpha)\frac{(\alpha,\alpha)}{2} h_\alpha = h_\alpha, \\ [h_\alpha, e_\alpha] &= \alpha(h_\alpha) e_\alpha = 2e_\alpha, \\ [h_\alpha, f_\alpha] &= -\alpha(h_\alpha) f_\alpha = -2f_\alpha. \end{aligned}$$

These relations justify defining $\mathfrak{sl}(2,\mathbb{C})_\alpha = \langle e_\alpha, f_\alpha, h_\alpha \rangle \subseteq \mathfrak{g}$.
This is a Lie subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

Brief review: irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$

Recall that an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ looks like

$$V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n.$$

Pictorially (with the weights in red):

$$\begin{tikzcd}
v_0 \arrow[r, bend left, "f"] \arrow[d, dash] & v_1 \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] \arrow[d, dash] & v_2 \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] \arrow[d, dash] & \cdots \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] & v_{n-1} \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] \arrow[d, dash] & v_n \arrow[l, bend left, "e"] \arrow[d, dash] \\
\color{red}{n} & \color{red}{n-2} & \color{red}{n-4} & & \color{red}{-n+2} & \color{red}{-n}
\end{tikzcd}$$

The weight spaces $V[n-2k] = \mathbb{C}v_k$ are all one-dimensional.

Finding representations of $\mathfrak{sl}(2,\mathbb{C})$

Lemma

For any root $\alpha \in R$, the subspace

$$V = \mathbb{C}h_\alpha \oplus \bigoplus_{k\in\mathbb{Z},\, k\neq 0} \mathfrak{g}_{k\alpha} \subseteq \mathfrak{g}$$

is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha = \langle e_\alpha, f_\alpha, h_\alpha\rangle$.

- We have $\operatorname{ad} e_\alpha.\mathfrak{g}_{k\alpha} \subseteq \mathfrak{g}_{(k+1)\alpha}$ and $\operatorname{ad} e_\alpha.\mathfrak{g}_{-\alpha} \subseteq \mathbb{C}h_\alpha$.
- Likewise $\operatorname{ad} f_\alpha.\mathfrak{g}_{k\alpha} \subseteq \mathfrak{g}_{(k-1)\alpha}$ and $\operatorname{ad} f_\alpha.\mathfrak{g}_{\alpha} \subseteq \mathbb{C}h_\alpha$.
- Therefore V is a representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$.
- Since $\alpha(h_\alpha) = 2$, all eigenvalues of $\operatorname{ad} h_\alpha$ are even:

$$V[2k] = \mathfrak{g}_{k\alpha} \quad \text{and} \quad V[0] = \mathbb{C}h_\alpha$$

- As $\dim V[0] = 1$, the representation is irreducible. $\square$

Main theorem about semisimple Lie algebras

We can now prove the main theorem about the structure of semisimple complex Lie algebras.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, with Cartan subalgebra $\mathfrak{h}$ and root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Let $(-,-)$ be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$; it induces nondegenerate pairings on $\mathfrak{h}$ and $\mathfrak{h}^*$. For every root $\alpha \in R$, we have a unique element $H_\alpha \in \mathfrak{h}$ with

$$(H_\alpha, h) = \alpha(h) \quad \text{for all } h \in \mathfrak{h}.$$

We defined $h_\alpha = \dfrac{2}{(\alpha, \alpha)} H_\alpha \in \mathfrak{h}$.

Main theorem about semisimple Lie algebras

Theorem

1. *The root system R spans $\mathfrak{h}^*$ as a vector space.*
2. *For each root $\alpha \in R$, we have $\dim \mathfrak{g}_\alpha = 1$.*
3. *For any $\alpha, \beta \in R$, we have $\beta(h_\alpha) = 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$.*
4. *For $\alpha \in R$, define the reflection operator $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by*

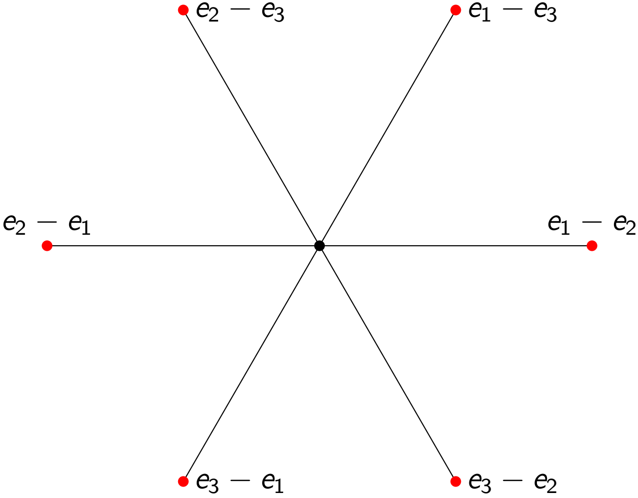
$$s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha.$$

   Then if $\beta \in R$, we also have $s_\alpha(\beta) \in R$.

5. *The only multiples of $\alpha \in R$ that are also roots are $\pm\alpha$.*
6. *If $\alpha, \beta \in R$ and $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Example: the root system of $\mathfrak{sl}(3, \mathbb{C})$

Here is the root system of $\mathfrak{sl}(3, \mathbb{C})$ from last time:



Proof of the main theorem, Part 1

1. The root system R spans $\mathfrak{h}^*$ as a vector space.

- Let $h \in \mathfrak{h}$ be such that $\alpha(h) = 0$ for every root $\alpha \in R$.
- The root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

  implies that $\operatorname{ad} h = 0$.
- But $\operatorname{ad} \colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ is injective, and so $h = 0$.
- This proves that R spans the dual vector space $\mathfrak{h}^*$. $\square$

Proof of the main theorem, Part 2

2. For each root $\alpha \in R$, we have $\dim \mathfrak{g}_\alpha = 1$.

- We proved that the subspace

$$V = \mathbb{C}h_\alpha \oplus \bigoplus_{k \in \mathbb{Z},\, k \neq 0} \mathfrak{g}_{k\alpha} \subseteq \mathfrak{g}$$

  is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$.
- Each weight subspace $V[2k] = \mathfrak{g}_{k\alpha}$ is one-dimensional.
- In particular, $\dim \mathfrak{g}_\alpha = \dim V[2] = 1$. $\square$

Proof of the main theorem, Part 3

3. For any $\alpha, \beta \in R$, we have $\beta(h_\alpha) = 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$.

- Consider $\mathfrak{g}$ as a representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$.
- With respect to $\operatorname{ad} h_\alpha$, the subspace $\mathfrak{g}_\beta$ has weight

$$\beta(h_\alpha) = \frac{2}{(\alpha, \alpha)}\beta(H_\alpha) = \frac{2}{(\alpha, \alpha)}(\alpha, \beta) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}.$$

- But we know that the weights in any finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ are integers. $\square$

Proof of the main theorem, Part 4

4. For $\alpha \in R$, define the reflection operator $s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha.$$

Then if $\beta \in R$, we also have $s_\alpha(\beta) \in R$.

- Suppose that $n = \beta(h_\alpha) \geq 0$.
- The subspace $\mathfrak{g}_\beta$ has weight n with respect to the representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$ on $\mathfrak{g}$.
- The operator $(\operatorname{ad} f_\alpha)^n$ gives an isomorphism between the subspace of weight n and the subspace of weight $-n$.
- Thus $x \in \mathfrak{g}_\beta$ nonzero implies $(\operatorname{ad} f_\alpha)^n x \in \mathfrak{g}_{\beta - n\alpha}$ nonzero.
- Consequently, $\beta - n\alpha = s_\alpha(\beta) \in R$.
- The proof in the other case $n \leq 0$ is similar.

Proof of the main theorem, Part 5

5. The only multiples of $\alpha \in R$ that are also roots are $\pm\alpha$.

- Suppose that $\beta = c\alpha \in R$ for some $c \in \mathbb{C}$.
- We know from Part 3 that

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2c \quad \text{and} \quad \frac{2(\alpha,\beta)}{(\beta,\beta))} = \frac{2}{c}$$

  are both integers. Therefore $c \in \{\pm 1, \pm 2, \pm\frac{1}{2}\}$.
- We can assume $c \in \{\pm 1, \pm 2\}$, by swapping α and β.
- We proved that the subspace

$$V = \mathbb{C}h_\alpha \oplus \bigoplus_{k\in\mathbb{Z},\, k\neq 0} \mathfrak{g}_{k\alpha} \subseteq \mathfrak{g}$$

  is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha = \langle e_\alpha, f_\alpha, h_\alpha \rangle$.
- But V contains $\mathfrak{sl}(2,\mathbb{C})_\alpha$, and so $V = \mathfrak{sl}(2,\mathbb{C})_\alpha$.
- Therefore $\mathfrak{g}_{2\alpha} = \mathfrak{g}_{-2\alpha} = 0$, hence $c \in \{\pm 1\}$. $\square$

Proof of the main theorem, Part 6

6. If $\alpha, \beta \in R$ and $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

- We already know that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
- We also know that $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$ for every root $\alpha \in R$.
- Therefore it is enough to show that

$$\mathfrak{g}_{\alpha+\beta} \neq 0 \quad \implies \quad [e_\alpha, e_\beta] \neq 0.$$

- For dimension reasons, the subspace

$$\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$$

  is again an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$.
- Therefore $\operatorname{ad} e_\alpha \colon \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+\alpha}$ is an isomorphism. $\square$

Positivity of the Killing form

Theorem

Let $\mathfrak{h}_{\mathbb{R}} \subseteq \mathfrak{h}$ be the real vector space spanned by $\{h_\alpha\}_{\alpha \in R}$.

1. *One has $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$.*
2. *The Killing form is positive definite on $\mathfrak{h}_{\mathbb{R}}$.*

Step 1: The Killing form is real valued on $\mathfrak{h}_{\mathbb{R}}$.

- Because of the root decomposition, we have

$$K(h_\alpha, h_\beta) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\, h_\alpha \circ \mathrm{ad}\, h_\beta) = \sum_{\gamma \in R} \gamma(h_\alpha)\gamma(h_\beta).$$

- But $\gamma(h_\alpha), \gamma(h_\beta) \in \mathbb{Z}$, and therefore $K(h_\alpha, h_\beta) \in \mathbb{Z}$.
- Since K is bilinear, it only takes real values on $\mathfrak{h}_{\mathbb{R}}$.

Positivity of the Killing form

Theorem

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1. *One has $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$.*
2. *The Killing form is positive definite on $\mathfrak{h}_{\mathbb{R}}$.*

Step 2: The Killing form is positive definite on $\mathfrak{h}_{\mathbb{R}}$.

- Let $h = \sum c_\alpha h_\alpha \in \mathfrak{h}_{\mathbb{R}}$.
- Then $\gamma(h) = \sum c_\alpha \gamma(h_\alpha) \in \mathbb{R}$ for every $\gamma \in R$.
- Again because of the root decomposition,

$$K(h,h) = \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad} h \circ \operatorname{ad} h) = \sum_{\gamma \in R} \gamma(h)^2 \geq 0.$$

- Since R spans $\mathfrak{h}^*$, we conclude that K is positive definite.

Positivity of the Killing form

Theorem

Let $\mathfrak{h}_{\mathbb{R}} \subseteq \mathfrak{h}$ be the real vector space spanned by $\{h_\alpha\}_{\alpha \in R}$.

1. *One has $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$.*
2. *The Killing form is positive definite on $\mathfrak{h}_{\mathbb{R}}$.*

Step 3: One has $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$.

- We know that the elements h_α span $\mathfrak{h}$.
- Therefore $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} + i\mathfrak{h}_{\mathbb{R}}$.
- From Step 2, K is positive definite on $\mathfrak{h}_{\mathbb{R}}$.
- Therefore K is negative definite on $i\mathfrak{h}_{\mathbb{R}}$.
- This gives $\mathfrak{h}_{\mathbb{R}} \cap i\mathfrak{h}_{\mathbb{R}} = \{0\}$, hence the result.