Cartan subalgebras and root decomposition

MAT 552

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Brief review

Today, $\mathfrak{g}$ is always a semisimple complex Lie algebra.

Last time, we defined semisimple and nilpotent elements:

- $x \in \mathfrak{g}$ is semisimple if $\operatorname{ad} x \in \operatorname{End}(\mathfrak{g})$ is semisimple.
- $x \in \mathfrak{g}$ is nilpotent if $\operatorname{ad} x \in \operatorname{End}(\mathfrak{g})$ is nilpotent.

For semisimple $\mathfrak{g}$, we proved that every $x \in \mathfrak{g}$ decomposes as

$$x = x_s + x_n$$

with x_s semisimple and x_n nilpotent. This decomposition is unique, and is called the (generalized) Jordan decomposition.

Example

In $\mathfrak{sl}(2, \mathbb{C})$, the element h is semisimple, and e, f are nilpotent.

Toral subalgebras

Definition

A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called toral if it is commutative and every element $h \in \mathfrak{h}$ is semisimple in $\mathfrak{g}$.

Example

Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, and let $\mathfrak{h}$ be the subalgebra of diagonal matrices with trace 0. Then $\mathfrak{h}$ is a toral subalgebra.

$$\begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} \xrightarrow{\exp} \begin{pmatrix} e^{h_1} & & & \\ & e^{h_2} & & \\ & & \ddots & \\ & & & e^{h_n} \end{pmatrix}$$

Here $\exp(\mathfrak{h}) \subseteq \mathrm{SL}(n, \mathbb{C})$ is the subgroup of diagonal matrices. It is an algebraic torus ($=$ a product of copies of $\mathbb{C}^*$).

Toral subalgebras

Definition

A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called toral if it is commutative and every element $h \in \mathfrak{h}$ is semisimple in $\mathfrak{g}$.

From linear algebra, we know that commuting semisimple elements can be diagonalized simultaneously:

- Suppose $x \in \mathfrak{g}$ is a common eigenvector for $\operatorname{ad} h$, $h \in \mathfrak{h}$.
- Then $\operatorname{ad} h.x = \alpha(h)x$, where $\alpha(h) \in \mathbb{C}$ is the eigenvalue.
- $\alpha\colon \mathfrak{h} \to \mathbb{C}$ is a linear functional, so $\alpha \in \mathfrak{h}^*$.

This gives us an eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; \operatorname{ad} h.x = \alpha(h)x \text{ for } h \in \mathfrak{h} \right\}$.

Toral subalgebras

Theorem

Let $\mathfrak{h} \subseteq \mathfrak{g}$ *be a toral subalgebra, and* $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$*. Then*

1. $\mathfrak{g}_0$ *is the centralizer* $C(\mathfrak{h})$*.*
2. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
3. *If* $x \in \mathfrak{g}_0$*, then* $x_s \in \mathfrak{g}_0$ *and* $x_n \in \mathfrak{g}_0$*.*
4. *If* $\alpha + \beta \neq 0$*, then* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_\beta$ *are orthogonal with respect to the Killing form* K*.*
5. K *induces nondegenerate pairings* $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$*.*
6. *The restriction of* K *to* $\mathfrak{g}_0$ *is nondegenerate.*
7. $\mathfrak{g}_0$ *is a reductive Lie algebra.*

Toral subalgebras

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a toral subalgebra, and $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$.

Proof that $[g_\alpha, g_\beta] \subseteq g_{\alpha+\beta}$:

- Take $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $h \in \mathfrak{h}$. Then

$$\begin{aligned} \operatorname{ad} h.[x, y] &= [h, [x, y]] \\ &= [[h, x], y] + [x, [h, y]] \\ &= [\alpha(h)x, y] + [x, \beta(h)y] \\ &= (\alpha(h) + \beta(h))[x, y]. \end{aligned}$$

- This says that $[x, y] \in \mathfrak{g}_{\alpha+\beta}$.

Proof that $x \in \mathfrak{g}_0$ implies $x_s, x_n \in \mathfrak{g}_0$:

- We have $[h, x] = 0$ for all $h \in \mathfrak{h}$.
- Therefore $[h, x_s] = 0$, hence $x_s \in \mathfrak{g}_0$ and $x_n = x - x_s \in \mathfrak{g}_0$.

Toral subalgebras

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a toral subalgebra, and $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$.

Proof that $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$:

- Take $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $h \in \mathfrak{h}$.
- Since K is invariant, we have

$$\begin{aligned} 0 &= K([h,x],y) + K(x,[h,y]) \\ &= K(\alpha(h)x, y) + K(x, \beta(h)y) \\ &= (\alpha(h) + \beta(h))K(x,y). \end{aligned}$$

- If $\alpha + \beta \neq 0$, this forces $K(x,y) = 0$.

Toral subalgebras

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a toral subalgebra, and $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$.

Proof that $\mathfrak{g}_0$ is reductive:

- K is nondegenerate (because $\mathfrak{g}$ is semisimple).
- We already know that $K(\mathfrak{g}_0, \mathfrak{g}_\alpha) = 0$ unless $\alpha = 0$.
- Therefore $K\colon \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathbb{C}$ must be nondegenerate.
- Consider $\mathfrak{g}$ as a representation of $\mathfrak{g}_0$.
- Since $K(x, y) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$, the trace pairing of this representation is nondegenerate.
- By an earlier theorem, this implies that $\mathfrak{g}_0$ is reductive.

$\square$

Cartan subalgebras

Find as many commuting semisimple elements as possible.

Definition

A Cartan subalgebra of a semisimple complex Lie algebra $\mathfrak{g}$ is a toral subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that

$$C(\mathfrak{h}) = \left\{ x \in \mathfrak{g} \;\middle|\; [x, h] = 0 \text{ for all } h \in \mathfrak{h} \right\} = \mathfrak{h}.$$

Basic facts:

1. Every maximal toral subalgebra is a Cartan subalgebra. In particular, Cartan subalgebras always exist.
2. All Cartan subalgebras are conjugate (under the adjoint action of the corresponding Lie group).
3. If $h \in \mathfrak{g}$ is a semisimple element with distinct eigenvalues, then $C(h)$ is a Cartan subalgebra.

Cartan subalgebras

Example

Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, and let $\mathfrak{h}$ be the subalgebra of diagonal matrices with trace 0. Then $\mathfrak{h}$ is a Cartan subalgebra.

- Every $h \in \mathfrak{h}$ is semisimple.
- $\mathfrak{h}$ is commutative, hence toral.
- Pick $h \in \mathfrak{h}$ with n distinct eigenvalues.
- If $[h, x] = 0$ for some $x \in \mathfrak{sl}(n, \mathbb{C})$, then every eigenvector of h is also an eigenvector of x, so x is also diagonal.
- Therefore
$$\mathfrak{h} \subseteq C(\mathfrak{h}) \subseteq C(h) = \mathfrak{h},$$
and so $\mathfrak{h}$ is a Cartan subalgebra.

Cartan subalgebras

Theorem

Every maximal toral subalgebra is a Cartan subalgebra.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a toral subalgebra that is not contained in any larger toral subalgebra. We need to show that $C(\mathfrak{h}) = \mathfrak{h}$.

As above, let

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

be the decomposition into eigenspaces. We shall argue that $\mathfrak{g}_0 = C(\mathfrak{h})$ is toral; by maximality, this implies $C(\mathfrak{h}) = \mathfrak{h}$.

Cartan subalgebras

We first argue that $\mathfrak{g}_0 = C(\mathfrak{h})$ is commutative:

- Let $x \in \mathfrak{g}_0$, with Jordan decomposition $x = x_s + x_n$.
- We already know that $x_s, x_n \in \mathfrak{g}_0$.
- Since x_s is semisimple, and $[h, x_s] = 0$ for all $h \in \mathfrak{h}$, the subalgebra $\mathfrak{h} \oplus \mathbb{C}x_s$ is still toral.
- By maximality of $\mathfrak{h}$, we must have $x_s \in \mathfrak{h}$.
- In particular, $\operatorname{ad} x_s|_{\mathfrak{g}_0} = 0$.
- It follows that $\operatorname{ad} x|_{\mathfrak{g}_0} = \operatorname{ad} x_n|_{\mathfrak{g}_0}$ is nilpotent.
- By Engel's theorem, $\mathfrak{g}_0$ is a nilpotent Lie algebra.
- Since $\mathfrak{g}_0$ is also reductive, it must be commutative.

Cartan subalgebras

Next, we argue that $\mathfrak{g}_0 = C(\mathfrak{h})$ is toral:

- We already know that $\mathfrak{g}_0$ is commutative.
- It remains to show that every $x \in \mathfrak{g}_0$ is semisimple.
- Consider the Jordan decomposition $x = x_s + x_n$.
- For any $y \in \mathfrak{g}_0$, the product $\operatorname{ad} x_n \circ \operatorname{ad} y$ is nilpotent (because $\operatorname{ad} x_n$ is nilpotent and $\mathfrak{g}_0$ is commutative).
- Therefore $K(x_n, y) = \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad} x_n \circ \operatorname{ad} y) = 0$.
- We showed earlier that the restriction of K to $\mathfrak{g}_0$ is nondegenerate; therefore $x_n = 0$.
- It follows that $x = x_s$ is semisimple.

We now conclude by maximality of $\mathfrak{h}$ that $C(\mathfrak{h}) = \mathfrak{h}$. $\square$

Cartan subalgebras

Basic facts:

1. Every maximal toral subalgebra is a Cartan subalgebra. In particular, Cartan subalgebras always exist.
2. All Cartan subalgebras are conjugate (under the adjoint action of the corresponding Lie group).
3. If $h \in \mathfrak{g}$ is a semisimple element with distinct eigenvalues, then $C(h)$ is a Cartan subalgebra.

In particular, all Cartan subalgebras have the same dimension. This dimension is called the rank of $\mathfrak{g}$.

Example

The rank of $\mathfrak{sl}(n, \mathbb{C})$ is $n - 1$.

Root decomposition

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. The eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

is called the root decomposition. The eigenspaces

$$\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \right\},$$

for nonzero $\alpha \in \mathfrak{h}^*$ are called the root subspaces. The set

$$R = \left\{ \alpha \in \mathfrak{h}^* \;\middle|\; \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0 \right\}$$

is called the root system of $\mathfrak{g}$. (Recall that $\mathfrak{g}_0 = \mathfrak{h}$.)

Root decomposition

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. The root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

has the following properties, proved earlier today:

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
2. If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the Killing form K.
3. For any α, the Killing form K induces a nondegenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$.
4. In particular, the restriction of K to $\mathfrak{h}$ is nondegenerate.

Example: $\mathfrak{sl}(n, \mathbb{C})$

Let us work out the example $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, with $\mathfrak{h}$ being the subalgebra of diagonal matrices of trace 0.

The i-th diagonal entry gives a linear functional

$$e_i \colon \mathfrak{h} \to \mathbb{C}, \quad \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} \mapsto h_i,$$

and clearly $e_1 + \cdots + e_n = 0$.

Therefore

$$\mathfrak{h}^* \cong \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \Big/ \mathbb{C}(e_1 + \cdots + e_n).$$

Example: $\mathfrak{sl}(n, \mathbb{C})$

Recall the following matrices:

$$E_{i,j} = \begin{array}{cl} \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} & \leftarrow i\text{-th row} \\ \uparrow & \\ j\text{-th column} & \end{array}$$

We have

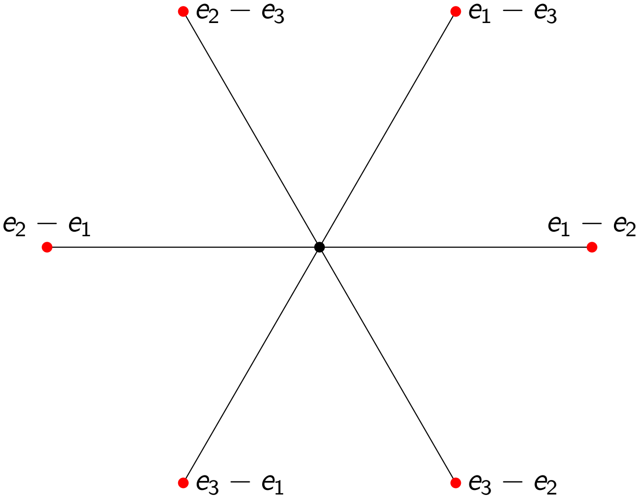
$$[h, E_{i,j}] = (h_i - h_j)E_{i,j} = (e_i(h) - e_j(h)) \cdot E_{i,j},$$

and so the root subspaces are exactly $\mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{i,j}$.

The root system is $R = \left\{ e_i - e_j \;\middle|\; i \neq j \right\} \subseteq \mathfrak{h}^*$.

Example: $\mathfrak{sl}(n, \mathbb{C})$

Here is a picture of the root system for $n = 3$:



Example: $\mathfrak{sl}(n, \mathbb{C})$

We can use the root decomposition to show that $\mathfrak{sl}(n, \mathbb{C})$ is a simple Lie algebra. (We did this earlier for $n = 2$.)

Let I be a nonzero ideal. I is stable under the action of $\operatorname{ad} h$, $h \in \mathfrak{h}$, and therefore generated by eigenvectors. Several cases:

1. If $h \in I$ for some $h \in \mathfrak{h}$, then

$$[h, E_{i,j}] = (e_i(h) - e_j(h)) \cdot E_{i,j}$$

   implies that $E_{i,j} \in I$ for some $i \neq j$.

2. If $E_{i,j} \in I$ for some $i \neq j$, then

$$[E_{i,j}, E_{k,\ell}] = \delta_{j,k} E_{i,\ell} - \delta_{i,\ell} E_{k,j}$$

   implies that $E_{i,j} \in I$ for all $i \neq j$.

3. Then $E_{i,i} - E_{j,j} \in I$ for all $i \neq j$, and so $I = \mathfrak{sl}(n, \mathbb{C})$.

Example: $\mathfrak{sl}(n, \mathbb{C})$

We can use this to show that $K(x, y) = 2n\,\mathrm{tr}(xy)$. (HW 4)

Since $\mathfrak{sl}(n, \mathbb{C})$ is simple, any two invariant bilinear forms are proportional. The Killing form must therefore be a multiple of the trace pairing. To find the constant, we compute K on $\mathfrak{h}$.

If $h \in \mathfrak{h}$, then

$$(\mathrm{ad}\,h)(E_{i,j}) = [h, E_{i,j}] = (h_i - h_j)E_{i,j},$$

and therefore

$$(\mathrm{ad}\,h \circ \mathrm{ad}\,h')(E_{i,j}) = (h_i - h_j)(h'_i - h'_j)E_{i,j}.$$

This gives

$$K(h, h') = \sum_{i \neq j}(h_i - h_j)(h'_i - h'_j) = 2n \sum_i h_i h'_i = 2n\,\mathrm{tr}(hh').$$