MATH 546, HOMEWORK 6

These problems are from Krantz, *Function Theory of Several Complex Variables.*

1. For a domain $D \subseteq \mathbb{C}^n$, denote by $P(D)$ the space of plurisubharmonic functions on D, and by $\mathscr{O}(D)$ the space of holomorphic functions. Suppose that D is pseudoconvex, and that $K \subseteq D$ is a compact. Prove that

$$\hat{K}_{P(D)} = \hat{K}_{\mathscr{O}(D)}$$

2. A Hartogs domain in $\mathbb{C}^n$ is a domain of the form

$$\left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; |z_n| < |f(z_1, \ldots, z_{n-1})| \right\}$$

where f is holomorphic. Prove that every Hartogs domain is a domain of holomorphy.

3. Let $D \subseteq \mathbb{C}^n$ be a domain of holomorphy. Let $\delta > 0$, and let $f_1, \ldots, f_k$ be holomorphic functions on D such that

$$\sum_{j=1}^{k} |f_j(z)| \geq \delta$$

for all $z \in D$. Prove that there are holomorphic functions $g_1, \ldots, g_k$ on D such that

$$\sum_{j=1^k} g_j f_j \equiv 1$$

on D. *Hint:* Consider the open covering of D by the sets

$$U_j = \left\{ z \in D \;\middle|\; |f_j(z)| > \delta/(2k) \right\}.$$

Let $\rho_1 + \cdots + \rho_k = 1$ be a partition of unity subordinate to this covering. Look for a solution to the problem of the form

$$g_j = \frac{\rho_j}{f_j} + \sum_{i=1}^{k} u_{j,i} f_i$$

where the $u_{j,i}$ are unknown functions that satisfy $u_{i,j} = -u_{j,i}$; then solve a suitable $\bar{\partial}$-equation to produce the $u_{j,i}$.

4. Let $D \subseteq \mathbb{C}^n$ be a domain of holomorphy. We say that D is *Runge* if each holomorphic function on D can be approximated by polynomials, uniformly on compact subsets of D. Let $\mathscr{O} = \mathscr{O}(D)$ denote the space of holomorphic functions on D, and $P = P(D)$ the space of polynomial functions. Show that the following four statements are equivalent:

(a) D is Runge.
(b) For every compact $K \subseteq D$, one has $\hat{K}_P = \hat{K}_{\mathscr{O}}$.
(c) For every compact $K \subseteq D$, one has $\hat{K}_P \cap D = \hat{K}_{\mathscr{O}}$
(d) For every compact $K \subseteq D$, one has $\hat{K}_P \cap D \subset\subset D$.

Due on Tuesday, November 24.