MATH 545—HOMEWORK 8

1. Curvature. The purpose of this exercise is to prove two assertions from class.

(a) Let L be a holomorphic line bundle with Hermitian metric h. Show that the dual line bundle inherits a Hermitian metric, whose curvature form is given by $\Theta_{L^{-1}} = -\Theta_L$.

(b) Let (L_1, h_1) and (L_2, h_2) be holomorphic line bundles with Hermitian metrics. Show that $L = L_1 \otimes L_2$ inherits a Hermitian metric, whose curvature form is given by $\Theta_L = \Theta_{L_1} + \Theta_{L_2}$.

2. The zero section of a line bundle. Let $\pi \colon L \to M$ be a holomorphic line bundle on M, and denote by $D \subseteq L$ the image of the zero section, a complex submanifold of codimension one. Prove that the line bundle $\mathscr{O}_L(-D)$ is isomorphic to $\pi^* L^{-1}$. (*Hint:* Show that a local trivialization $\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ of L induces a local trivialization of $\mathscr{O}_L(-D)$ on the open set $V_\alpha = \pi^{-1}(U_\alpha)$, and compare the transition functions.)

3. Embeddings into projective space. Consider the holomorphic mapping $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^k$ defined by the line bundle $\mathscr{O}_{\mathbb{P}^1}(k)$; using the monomials of degree k as a basis, it is given by the formula $[z_0, z_1] \mapsto \left[z_0^k, z_0^{k-1} z_1, \dots, z_0 z_1^{k-1}, z_1^k\right]$. Prove that the image is a complex submanifold of $\mathbb{P}^k$, and that φ is a biholomorphism between $\mathbb{P}^1$ and the image.

4. Holomorphic sections. Let L be a holomorphic line bundle on a compact manifold M. Prove that any global holomorphic section of L is $\bar\partial$-harmonic.

5. Serre duality. The goal of this exercise is to prove the Serre duality theorem for holomorphic line bundles on a compact complex manifold: $H^q(M, \Omega_M^p \otimes L)$ and $H^{n-q}(M, \Omega_M^{n-p} \otimes L^{-1})$ are dual vector spaces.

(a) Fix a Hermitian metric on L, and give the dual line bundle L^{-1} the induced metric. Reduce the theorem to showing that the harmonic spaces $\mathcal{H}^{p,q}(M, L)$ and $\mathcal{H}^{n-p,n-q}(M, L^{-1})$ are dual vector spaces.

(b) Show that a Hermitian inner product h on a complex vector space V induces a conjugate-linear isomorphism $V \to \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ by the rule $v \mapsto h(v, -)$.

(c) Locally, any section of $A^{p,q}(M, L)$ can be written in the form $\alpha \otimes s$, with $\alpha \in A^{p,q}(U)$ and $s \in A(U, L)$ holomorphic. Show that $\sharp(\alpha \otimes s) = (*\overline{\alpha}) \otimes h(s, -)$ does not depend on the choice of α and s, and is a well-defined conjugate-linear operator from $A^{p,q}(M, L)$ to $A^{n-p,n-q}(M, L^{-1})$.

(d) Show that the rule $(\alpha \otimes s) \wedge (\beta \otimes \phi) = \phi(s)(\alpha \wedge \beta)$ gives a well-defined bilinear mapping $\wedge \colon A^{p,q}(M, L) \times A^{n-p,n-q}(M, L^{-1}) \to A^{n,n}(M)$.

(e) In class, we introduced a Hermitian inner product on the space $A^{p,q}(M, L)$. Show that it satisfies $(\alpha, \beta)_L = \int_M \alpha \wedge \sharp\beta$.

(f) Prove the formula $\bar\partial^* = -\sharp\bar\partial\sharp$ for the adjoint of $\bar\partial$, and deduce that $\sharp$ commutes with the Laplace operator $\overline{\Box}$. Deduce the Serre duality theorem.

Due on Thursday, November 21.