MATH 545—HOMEWORK 6

1. The Levi-Cività connection. Let $\mathbb{S}^2$ be the two-sphere, consisting of all points $(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 + z^2 = 1$. As a submanifold of $\mathbb{R}^3$, it inherits a Riemannian metric g.

(a) Let H be the plane $z = -1$, with coordinates x_1, x_2. Stereographic projection from the point $(0, 0, 1)$ to H defines a natural coordinate chart on $\mathbb{S}^2$. Compute the coefficients $g_{i,j}$ for the metric in these coordinates.

(b) In the same chart, compute the Levi-Cività connection ∇; in other words, find formulas for the $\Gamma^k_{i,j}$.

(c) Tangent vectors to $\mathbb{S}^2$ are just vectors in $\mathbb{R}^3$. Can you find an interpretation for $\nabla_\xi \eta$ in terms of vectors in $\mathbb{R}^3$?

(d) Since $H \simeq \mathbb{C}$, we get a complex structure on $\mathbb{S}^2$ by setting $J\partial/\partial x_1 = \partial/\partial x_2$ and $J\partial/\partial x_2 = -\partial/\partial x_1$. Describe the action of J in terms of vectors in $\mathbb{R}^3$.

2. Kähler metrics. Let M be a Riemann surface, and h a Hermitian metric on it. Show that h is a Kähler metric.

3. Complex tori. Recall that a complex torus is a quotient $M = \mathbb{C}^n/\Gamma$, where Γ is a lattice in $\mathbb{C}^n$, that is, a discrete subgroup of rank $2n$.

(a) Show that the standard metric on $\mathbb{C}^n$ induces a Hermitian metric on M.

(b) Prove that this metric is Kähler.

4. Riemann surfaces. Let M be a compact Riemann surface, and let h be a Hermitian metric on it. As usual, $g = \operatorname{Re} h$ will denote the induced Riemannian metric on the underlying two-manifold.

(a) Let $z = x + iy$ be a holomorphic coordinate on an open subset $U \subseteq M$. Show that $\phi = h(\partial/\partial z, \partial/\partial z)$ is a real-valued smooth function on U, and that $vol(g) = \phi \cdot dx \wedge dy$.

(b) Prove that $*dx = dy$, $*dy = -dx$, and therefore $*dz = -idz$ and $*d\bar{z} = id\bar{z}$. Conclude that the $*$-operator preserves the space $H^0(M, \Omega^1_M)$ of holomorphic one-forms on M.

(c) Show that any $\omega \in H^0(M, \Omega^1_M)$ satisfies $d\omega = 0$.

(d) Conclude that $\Delta\omega = 0$, which means that ω is harmonic for the given metric.

Due on Thursday, October 31.