MATH 545—HOMEWORK 5

1. Cartan's lemma. Let $\mathbf{U}$ be an open cover of a complex manifold M, with the property that $H^{0,1}(U_i) \simeq 0$ for every $i \in I$. The purpose of this exercise is to prove a special case of Cartan's lemma: $H^1(\mathbf{U}, \mathscr{O}) \simeq H^{0,1}(M)$.

(a) Let $\sum_i \rho_i = 1$ be a partition of unity subordinate to the open cover. Given a cocycle $\mathbf{f} \in C^1(\mathbf{U}, \mathscr{O})$, set $\phi_j = \sum_i \rho_i f_{i,j} \in A(U_j)$. Show that $\bar{\partial}\phi_j$ is the restriction of a $\bar{\partial}$-closed form $\alpha(\mathbf{f}) \in A^{0,1}(M)$.

(b) Show that $\mathbf{f} \mapsto \alpha(\mathbf{f})$ defines a linear map $\alpha \colon H^1(\mathbf{U}, \mathscr{O}) \to H^{0,1}(M)$ from Čech cohomology to Dolbeault cohomology.

(c) Prove that α is an isomorphism.

2. Soft sheaves. Let $\mathscr{F}$ be a sheaf on a topological space X. Prove that the sheaf of discontinuous sections $\operatorname{ds} \mathscr{F}$ is a soft sheaf.

3. Harmonic forms. Let (M, g) be a compact oriented Riemannian manifold of dimension n. Show that $\big(\operatorname{vol}(g), d\psi\big)_M = \int_M d\psi$ for any $\psi \in A^{n-1}(M)$. Conclude that the volume form $\operatorname{vol}(g) \in A^n(M)$ is harmonic.

4. Inner products.

(a) Let (V_1, g_1) and (V_2, g_2) be two real vector spaces with inner products. Show that the formula

$$g(u_1 \otimes u_2, v_1 \otimes v_2) = g_1(u_1, v_1) g_2(u_2, v_2),$$

extended bilinearly, defines an inner product on $V_1 \otimes V_2$.

(b) Now let (V, g) be a real vector space with inner product, and give $V^{\otimes k}$ the inner product from (a). We can embed $\bigwedge^k V$ into $V^{\otimes k}$ by the rule

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma) \cdot v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

where the sum is over all permutations of $\{1, \dots, k\}$. Show that the induced inner product on $\bigwedge^k V$ agrees with the one defined in class.

5. The Laplace operator. Let (M, g) be a compact oriented Riemannian manifold, and consider the Laplacian $\Delta = d^* \circ d \colon A(M) \to A(M)$.

(a) In local coordinates $x_1, \dots, x_n$, let G be the matrix with entries $g_{i,j} = g(\partial/\partial x_i, \partial/\partial x_j)$. Show that we have $g(dx_i, dx_j) = g^{i,j}$, where $g^{i,j}$ are the entries of the inverse matrix G^{-1}.

(b) Prove the formula

$$\Delta f = -\sum_{i,j=1}^{n} \frac{1}{\sqrt{\det G}} \frac{\partial}{\partial x_i} \left(g^{i,j} \frac{\partial f}{\partial x_j} \sqrt{\det G} \right).$$

(Hint: Use the identity $\int_M \varphi \Delta f \, vol(g) = \int_M g(df, d\varphi) vol(g)$.)

(c) Compute the symbol $P(x, \xi)$, and conclude that Δ is elliptic of order 2.

Due on Tuesday, October 22.

6. Poincaré duality.

(a) Show that if $\omega \in A^k(M)$ is a harmonic form, then $*\omega$ is also harmonic.
(b) Show that we have an isomorphism $H^k(M, \mathbb{R}) \simeq H^{n-k}(M, \mathbb{R})$.
(c) Prove the Poincaré duality theorem, namely that the pairing

$$H^k(M, \mathbb{R}) \times H^{n-k}(M, \mathbb{R}) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$$

is nondegenerate in each argument. (The point is that this formulation does not involve the Riemannian metric.)

7. An orthogonal decomposition. Let $V_{\mathbb{R}}$ be a real vector space of dimension $2n$, with a complex structure $J \in \operatorname{End}(V_{\mathbb{R}})$ and a compatible inner product g; as in class, this means that $J^2 = -\operatorname{id}$, and that $g(Jv_1, Jv_2) = g(v_1, v_2)$. From g, we get an induced inner product on $\bigwedge^k V_{\mathbb{R}}$, which we can then extend sesquilinearly (= linearly in the first, conjugate linearly in the second argument) to a hermitian inner product h on $\bigwedge^k V_{\mathbb{C}}$. Prove that the decomposition $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$ is orthogonal with respect to h.