MATH 554—HOMEWORK 3

1. Quotients. Let X be a complex manifold, and $\Gamma \subseteq \operatorname{Aut}(X)$ a properly discontinuous group of automorphisms without fixed points.

(a) Show that $R = \{ (x,y) \in X \times X \mid x \sim y \}$ is a closed subset of $X \times X$.

(b) Deduce that X/Γ is Hausdorff and has a countable basis. (Hint: The Hausdorff property is equivalent to (a) because $q\colon X \to X/\Gamma$ is open.)

(c) Show that every point has an open neighborhood $U \subseteq X$ with the property that $\gamma(U) \cap U = \emptyset$ for $\gamma \in \Gamma$, $\gamma \neq \mathrm{id}$.

(d) Deduce that every point of X has an open neighborhood U with the following two properties: U is biholomorphic to an open subset of $\mathbb{C}^n$; and U is mapped homeomorphically to its image $q(U)$ in the quotient.

(e) Conclude that X/Γ is a complex manifold, and that q is holomorphic and locally biholomorphic.

2. Singular locus. Let $Z \subseteq D$ be an analytic subset of an open set $D \subseteq \mathbb{C}^n$, and suppose that $0 \in Z$, but that Z does not contain any open neighborhood of 0.

(a) Prove the following lemma: If $I \subseteq \mathscr{O}_n$ is a nonzero ideal such that $\partial f/\partial z_j \in I$ for every $f \in I$ and every $j = 1, \dots, n$, then $I = \mathscr{O}_n$. (Hint: Consider the smallest $d \geq 0$ for which there exists $f \in I$ regular of degree d.)

(b) Now let $k \geq 0$ be the largest integer such that there exist $f_1, \dots, f_k \in I(Z)$ with the property that at least one $k \times k$-minor g of the matrix $J(f)$ does not belong to $I(Z)$. Prove that $k \geq 1$.

(c) After shrinking D, we may assume that $f_1, \dots, f_k \in \mathscr{O}(D)$, and define $D' = D \setminus Z(g)$. Show that $Z' = Z(f_1) \cap \dots \cap Z(f_k) \cap D'$ is a submanifold of D', and that $Z \cap D'$ is a union of connected components of Z'.

(d) Conclude that the singular locus Z^s is contained in an analytic set strictly smaller than Z.

3. Plane curves.

(a) Let $F \in \mathbb{C}[z_0, z_1, z_2]$ be a homogeneous polynomial of degree $d \geq 1$, and let $Z(F) = \{ [z] \in \mathbb{P}^2 \mid F(z) = 0 \}$. Show that $Z(F)$ is a one-dimensional complex submanifold of $\mathbb{P}^2$, if and only if, at least one of the partial derivatives $\partial F/\partial z_j$ is nonzero at every point of $\mathbb{P}^2$.

(b) Determine all $a, b \in \mathbb{C}$ for which the equation $z_0 z_2^2 - z_1^3 - a z_0^2 z_1 - b z_0^3 = 0$ defines a submanifold of $\mathbb{P}^2$.

4. Blow-ups. Recall that we constructed $\operatorname{Bl}_0 \mathbb{C}^n$ as a subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$.

(a) Show that the second projection $q\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{P}^{n-1}$ is a holomorphic vector bundle of rank one, and compute its transition functions with respect to the open cover $U_1, \dots, U_n$ of $\mathbb{P}^{n-1}$.

(b) Prove that the line bundle in (a) does not have nontrivial global sections.

Due on Thursday, September 26.