MATH 545—HOMEWORK 2

1. Irreducibility.

(a) Let $f \in \mathscr{O}_n$. Show that if one of the partial derivatives $\partial f/\partial z_j(0)$ is nonzero, then f is irreducible in $\mathscr{O}_n$.

(b) More generally, let $f_1, \dots, f_m \in \mathscr{O}_n$, and suppose that the matrix $J(f) = \partial(f_1, \dots, f_m)/\partial(z_1, \dots, z_n)$ has rank m when $z = 0$. Show that the analytic set $Z(f)$ is irreducible in a neighborhood of the origin.

2. Irreducible decomposition. Consider the holomorphic function $f(z_1, z_2) = z_1^2 - z_2^2 - z_2^3$ on $\mathbb{C}^2$. Show that f is reducible in the local ring $\mathscr{O}_2$, and find the irreducible components of the analytic set $Z(f)$ in a neighborhood of the origin.

3. Inverse mapping theorem. Let $f\colon D \to \mathbb{C}^n$ be a holomorphic mapping, defined on an open neighborhood of $0 \in \mathbb{C}^n$. Prove that if $J(f)$ is nonsingular at the point 0, then f has a holomorphic inverse near $f(0)$.

4. Products of complex manifolds. Let X and Y be complex manifolds. Prove that their product $X \times Y$ is again a complex manifold, and that $\dim_{(x,y)} X \times Y = \dim_x X + \dim_y Y$.

5. Projective space.

(a) Compute the transition functions between the standard coordinate charts $U_0, U_1, \dots, U_n$ on $\mathbb{P}^n$.

(b) Prove that $\mathscr{O}_{\mathbb{P}^n}(\mathbb{P}^n) \simeq \mathbb{C}$.

6. Zero locus. Let f, g be holomorphic functions in some neighborhood of $0 \in \mathbb{C}^n$. Suppose that f is irreducible in $\mathscr{O}_n$, and that g vanishes on $Z(f)$, in the sense that $g(z) = 0$ for every $z \in Z(f)$. Prove that f divides g in $\mathscr{O}_n$.

Due on Tuesday, September 17.