MATH 545—HOMEWORK 1

1. Chain rule. Let $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$ be open subsets, and let $f\colon D \to E$ and $g\colon E \to \mathbb{C}$ be differentiable mappings. Prove the following version of the chain rule,

$$\begin{aligned}\frac{\partial(g \circ f)}{\partial \bar{z}_k} &= \sum_j \left(\frac{\partial g}{\partial w_j} \frac{\partial f_j}{\partial \bar{z}_k} + \frac{\partial g}{\partial \bar{w}_j} \frac{\partial \bar{f}_j}{\partial \bar{z}_k} \right) \\ \frac{\partial(g \circ f)}{\partial z_k} &= \sum_j \left(\frac{\partial g}{\partial w_j} \frac{\partial f_j}{\partial z_k} + \frac{\partial g}{\partial \bar{w}_j} \frac{\partial \bar{f}_j}{\partial z_k} \right),\end{aligned}$$

where $z_1, \dots, z_n$ are the coordinates on D, and $w_1, \dots, w_m$ the coordinates on E.

2. Zero sets.

(a) Let $f\colon D \to \mathbb{C}$ be a holomorphic function on $D \subseteq \mathbb{C}$, and assume that $\overline{\Delta}(a; r) \subseteq D$. Using Cauchy's formula, prove Jensen's inequality

$$\log|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log|f(a + re^{i\theta})| d\theta.$$

(b) Now let $f\colon D \to \mathbb{C}$ be a holomorphic function on $D \subseteq \mathbb{C}^n$. Show that for every polydisk $\overline{\Delta}(a; r) \subseteq D$, one has

$$\log|f(a)| \leq \frac{1}{\operatorname{vol}\bigl(\Delta(a; r)\bigr)} \int_{\Delta(a;r)} \log|f(z)| dz.$$

(c) Use (b) to prove that if D is connected and $f \in \mathscr{O}(D)$ does not vanish everywhere in D, then its zero set $Z(f) = f^{-1}(0)$ has measure zero.

3. Hartog's lemma. Let $R_j > r_j > 0$ for $j = 1, \dots, n$, and consider the domain $D = \Delta(0; R) \setminus \overline{\Delta}(0; r)$ obtained by removing a smaller closed polydisk from a larger open polydisk. If $n \geq 2$, prove that every holomorphic function $f\colon D \to \mathbb{C}$ can be uniquely extended to a holomorphic function on the whole polydisk $\Delta(0; R)$. Give an example to show that this statement is not true when $n = 1$.

4. Riemann's extension theorem. Let $D \subseteq \mathbb{C}^n$ be an open subset and $f \in \mathscr{O}(D)$ a holomorphic function. If $g\colon D - Z(f) \to \mathbb{C}$ is holomorphic and bounded, prove that it extends uniquely to a holomorphic function on all of D.

5. Hensel's lemma. Let $h(z, t) \in \mathscr{O}_n[t]$ be a monic polynomial of degree d, with coefficients in the ring $\mathscr{O}_n$. Then $h(0, t)$ is a monic polynomial in t, and can therefore be factored by the fundamental theorem of algebra as

$$h(0, t) = (t - c_1)^{d_1} \cdots (t - c_r)^{d_r}$$

where $d_1 + \cdots + d_r = d$ and the $c_j \in \mathbb{C}$ are distinct. Prove that there are uniquely determined monic polynomials $p_1, \dots, p_r \in \mathscr{O}_n[t]$, such that $h = p_1 \cdots p_r$ and $p_j(0, t) = (t - c_j)^{d_j}$ for every $j = 1, \dots, r$.

Due on Tuesday, September 10.