Class 8. The Poincaré lemma and integration (September 24)

The $\bar\partial$-Poincaré lemma. The key step in proving de Rham's theorem is to show that closed forms are always locally exact. The same result is true for Dolbeault cohomology, and is the content of the so-called $\bar\partial$-Poincaré lemma.

Lemma 8.1. *Let $D \subseteq \mathbb{C}^n$ be an open subset, and $\omega \in A^{p,q+1}(D)$ be a $\bar\partial$-closed form with $q \geq 0$. Then for any relatively compact open set U with $\bar U \subseteq D$, there is a (p,q)-form $\psi \in A^{p,q}(U)$ such that $\omega = \bar\partial\psi$ on U.*

As a warm-up, let us prove the $\bar\partial$-Poincaré lemma in one complex variable.

Lemma 8.2. *Let $g\colon \mathbb{C} \to \mathbb{C}$ be a smooth function with compact support. Then the (singular) integral*

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(w)}{w-z} dw \wedge d\bar w \tag{8.3}$$

converges for every $z \in \mathbb{C}$, and defines a smooth, compactly supported function with $\partial f/\partial\bar z = g$.

Proof. Recall the more precise form of Cauchy's formula: Let $D = \Delta(z;R)$ and $D_\varepsilon = \Delta(z;\varepsilon)$. If f is smooth in a neighborhood of the closed disk $\overline{D}$, then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}\int_D \frac{\partial f}{\partial \bar w}(w)\frac{dw\wedge d\bar w}{w-z}. \tag{8.4}$$

This is proved by letting $\alpha = (2\pi i)^{-1} f(w)dw/(w-z)$, and applying Stokes' theorem

$$\int_{D\setminus D_\varepsilon} d\alpha = \int_{\partial D}\alpha - \int_{\partial D_\varepsilon}\alpha.$$

to obtain the identity

$$-\frac{1}{2\pi i}\int_{D\setminus D_\varepsilon} \frac{\partial f}{\partial\bar w}\frac{dw\wedge d\bar w}{w-z} = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\partial D_\varepsilon}\frac{f(w)}{w-z}dw.$$

After setting $w = re^{i\theta} + z$ and computing that $dw \wedge d\bar w = 2ir \cdot d\theta \wedge dr$, this becomes

$$-\frac{1}{\pi}\int_{D\setminus D_\varepsilon} \frac{\partial f}{\partial \bar w}(z+re^{i\theta})\cdot e^{-i\theta} d\theta\wedge dr = \int_0^{2\pi} f(z+Re^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi} f(z+\varepsilon e^{i\theta})\frac{d\theta}{2\pi}$$

and converges to the asserted formula as $\varepsilon \to 0$, because the integrands are smooth functions.

We now prove the lemma. Changing to polar coordinates by again setting $w = re^{i\theta} + z$, the integral in (8.3) becomes

$$f(z) = \frac{1}{\pi}\int_{\mathbb{C}} g(z+re^{i\theta})\cdot e^{-i\theta} d\theta \wedge dr.$$

Since g has compact support, it is clear from this expression that f is well-defined and smooth on $\mathbb{C}$. Interchanging the order of differentiation and integration, and undoing the change of coordinates, we then have

$$\frac{\partial f}{\partial \bar z}(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g}{\partial \bar w}(z+re^{i\theta})\cdot e^{-i\theta} d\theta\wedge dr = \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\partial g}{\partial \bar w}(w)\frac{dw\wedge d\bar w}{w-z}.$$

Now the support of g is contained in $D = \Delta(z;R)$ for sufficiently large R, and so we get the result by applying (8.4), noting that the integral over ∂D is zero. □

We can now prove the higher-dimensional version of the $\bar\partial$-Poincaré lemma.

Proof of Lemma 8.1. The proof of the lemma works by induction; the k-th step is to show that the statement is true when ω does not depend on $d\bar{z}_{k+1}, \dots, d\bar{z}_n$. This is clearly trivial when $k = 0$, and gives us the desired result when $k = n$). Suppose that the statement has been proved for $k-1$, and that ω does not involve $d\bar{z}_{k+1}, \dots, d\bar{z}_n$. Write ω in the form $\alpha \wedge d\bar{z}_k + \beta$, where $\alpha \in A^{p,q}(\mathbb{C}^n)$ and $\beta \in A^{p,q+1}(\mathbb{C}^n)$ do not depend on $d\bar{z}_k, \dots, d\bar{z}_n$. As usual, let $\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J$; then $\bar{\partial}\omega = 0$ implies that $\partial \alpha_{I,J} / \partial \bar{z}_j = 0$ for every $j > k$.

Now choose a smooth function ρ with compact support inside D that is identically equal to 1 on an open neighborhood V of $\bar{U}$. By the above,

$$\varphi_{I,J}(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \alpha_{I,J}(z_1, \dots, z_{k-1}, w, z_{k+1}, \dots, z_n) \frac{\rho(w)}{w - z_k} dw \wedge d\bar{w}$$

is a smooth function on D; it satisfies $\partial \varphi_{I,J} / \partial \bar{z}_j = 0$ for $j > k$, and $\partial \varphi_{I,J} / \partial \bar{z}_k = \alpha_{I,J}$ at every point of V. If we now let $\varphi = \sum \varphi_{I,J} dz_I \wedge d\bar{z}_J$, then $\omega - \bar{\partial}\varphi$ is independent of $\bar{z}_k, \dots, \bar{z}_n$ on V. By induction, we can find $\psi' \in A^{p,q}(U)$ such that $\omega - \bar{\partial}\varphi = \bar{\partial}\psi'$, and then $\psi = \varphi + \psi'$ does the job. □

By writing any (generalized) polydisk as an increasing union of relatively compact polydisks, one can then deduce the following proposition.

Proposition 8.5. *Let $D = \{ z \in \mathbb{C}^n \mid |z_j| < r_j \}$, where we allow the possibility that some or all $r_j = \infty$. Then $H^{p,q}(D) = 0$ for $q \geq 1$.*

Integration. Differential forms are connected with integration on manifolds, as follows. Suppose that M is an oriented manifold, meaning that we have a consistent choice of orientation on each tangent space $T_{\mathbb{R},p}M$. It then makes sense to talk about the orientation of a system of local coordinates: $x_1, \dots, x_n$ is positively oriented if the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$ form a positive basis in each $T_{\mathbb{R},p}M$. (A necessary and sufficient condition for being orientable is that the transition functions $h_{\alpha,\beta}$ between local charts are orientation preserving, in the sense that $\det J_{\mathbb{R}}(h_{\alpha,\beta}) > 0$.)

Let $\omega \in A^n(M)$ be a smooth n-form with compact support. We can cover the support of ω by finitely many coordinate charts U_α, and choose a partition of unity $1 = \sum \rho_\alpha$ subordinate to the covering. In positively oriented local coordinates $x_{\alpha,1}, \dots, x_{\alpha,n}$, we have

$$(\rho_\alpha \omega)|_{U_\alpha} = \varphi_\alpha dx_{\alpha,1} \wedge \dots \wedge dx_{\alpha,n},$$

where φ_α are smooth functions with compact support in $D_\alpha \subseteq \mathbb{R}^n$. We then define the integral of ω over M by the formula

$$\int_M \omega = \sum_\alpha \int_{D_\alpha} \varphi_\alpha d\mu, \tag{8.6}$$

where μ is Lebesgue measure on $\mathbb{R}^n$. Note that this definition makes sense: by (7.8), we have

$$dx_{\alpha,1} \wedge \dots \wedge dx_{\alpha,n} = \left(\det J_{\mathbb{R}}(h_{\alpha,\beta}) \circ h_{\alpha,\beta}^{-1}\right) \cdot dx_{\beta,1} \wedge \dots \wedge dx_{\beta,n},$$

and since M is orientable, there is no problem with the choice of sign. It follows from the usual change of variables formula for integrals that the definition does not depend on the choice of coordinates.

As in calculus, Stokes' theorem is valid: if $\psi \in A^{n-1}(M)$ has compact support, then $\int_M d\psi = 0$. This proves the familiar fact that, on a compact orientable

n-dimensional manifold, $H^n(X, \mathbb{R}) \simeq \mathbb{R}$, where the isomorphism is given by integration over M.

An important fact in complex geometry is that any complex manifold M is automatically orientable. Indeed, the transition functions $h_{\alpha,\beta}$ between coordinate charts are now biholomorphic, and we have seen in (7.2) that $\det J_{\mathbb{R}}(h_{\alpha,\beta}) = |J(h_{\alpha,\beta})|^2 > 0$. We take the natural orientation to be the one given in local coordinates $z_j = x_j + iy_j$ by the ordering

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n.$$

We can therefore integrate any compactly supported form $\omega \in A^{n,n}(M)$, and the integral $\int_M \omega$ is a complex number. Noting that $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy$, we compute that

$$(dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n) = \frac{i^n}{2^n}(dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n);$$

this takes the place of Lebesgue measure in the definition of the integral above.