## Class 6. The tangent bundle (September 17)

Last time, we defined vector bundles as continuous maps $\pi\colon E \to M$ that are locally trivial.

**Definition 6.1.** A *section* of a vector bundle $\pi\colon E \to M$ over an open set $U \subseteq M$ is a continuous map $s\colon U \to E$ with the property that $\pi \circ s = \mathrm{id}_U$. We denote the set of all sections of $E$ over $U$ by the symbol $\Gamma(U, E)$.

When $E$ is a smooth (resp., holomorphic) vector bundle, we usually require sections to be smooth (resp., holomorphic). It is a simple matter to describe sections in terms of transition functions: Suppose we are given a section $s\colon M \to E$. For each local trivialization $\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^k$, the composition $\phi_\alpha \circ s$ is necessarily of the form $(\mathrm{id}, s_\alpha)$ for a continuous mapping $s_\alpha\colon U_\alpha \to \mathbb{K}^k$, and one checks that

$$g_{\alpha,\beta} \cdot s_\beta = s_\alpha \quad \text{on } U_\alpha \cap U_\beta. \tag{6.2}$$

Conversely, every collection of mappings $s_\alpha$ that satisfies these identities describes a section of $E$. Since (6.2) is clearly $\mathbb{K}$-linear, it follows that the set $\Gamma(U, E)$ is actually a $\mathbb{K}$-vector space.

**Tangent spaces and tangent bundles.** On a manifold, the most natural example of a vector bundle is the tangent bundle. Before discussing complex manifolds, we first review the basic properties of the tangent bundle on a smooth manifold.

Let $M$ be a smooth manifold; to simplify the discussion, we assume that $M$ is connected and let $n = \dim M$. Given any point $p \in M$, there is an isomorphism $f\colon U \to D$ between a neighborhood of $p$ and an open subset $D \subseteq \mathbb{R}^n$; we may clearly assume that $f(p) = 0$. By composing the coordinate functions $x_1, \dots, x_n$ on $\mathbb{R}^n$ with $f$, we obtain $n$ smooth functions on $U$; they form a *local coordinate system* around the point $p \in M$. Despite the minor ambiguity, we continue to denote the coordinate functions by $x_1, \dots, x_n \in \mathscr{A}_M(U)$. Note that we have $x_j(p) = 0$ for every $j$.

On $\mathbb{R}^n$, we have $n$ vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$ that act as derivations on the ring of smooth functions on $D$. By composing with $f$, we can view them as smooth vector fields on $U \subseteq M$; the action on $\mathscr{A}_M(U)$ is now given by the rule

$$\frac{\partial}{\partial x_j}\psi = \frac{\partial(\psi \circ f^{-1})}{x_j} \circ f$$

for any smooth function $\psi\colon U \to \mathbb{R}$. The values of those vector fields at the point $p$ give a basis for the *real tangent space*

$$T_{\mathbb{R},p}M = \mathbb{R}\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}.$$

The tangent bundle $T_\mathbb{R} M$ is the smooth vector bundle with fibers $T_{\mathbb{R},p}M$; its sections are smooth vector fields. To obtain transition functions for $T_\mathbb{R} M$, let us see how vector fields transform between coordinate charts. To simplify the notation, let $f\colon U \to D$ and $g\colon U \to E$ be two charts with the same domain; we denote the coordinates on $D$ by $x_1, \dots, x_n$, and the coordinates on $E$ by $y_1, \dots, y_n$. As usual, we let $h = f \circ g^{-1}\colon E \to D$ be the diffeomorphism that compares the two charts.

Now say

$$\sum_{j=1}^n a_j(x) \frac{\partial}{\partial x_j} \quad \text{and} \quad \sum_{k=1}^n b_k(y) \frac{\partial}{\partial y_k}$$

are smooth vector fields on $D$ and $E$, respectively, that represent the same vector field on $U$. Let $\psi\colon D \to \mathbb{R}$ be a smooth function; then since $\psi(x) = \psi\bigl(h(y)\bigr)$, we compute with the help of the chain rule that

$$\frac{\partial}{\partial y_k}\psi = \frac{\partial(\psi \circ h)}{\partial y_k} \circ h^{-1} = \sum_{j=1}^{n} \left( \frac{\partial h_j}{\partial y_k} \circ h^{-1} \right) \cdot \frac{\partial \psi}{\partial x_j}.$$

This means that, as vector fields on $D$,

$$\frac{\partial}{\partial y_k} = \sum_{j=1}^{n} \frac{\partial h_j}{\partial y_k}\bigl(h^{-1}(x)\bigr) \frac{\partial}{\partial x_j},$$

and so it follows that the coefficients in the two coordinate systems are related by the identity

$$a_j(x) = \sum_{k=1}^{n} \frac{\partial h_j}{\partial y_k}\bigl(h^{-1}(x)\bigr) \cdot b_k\bigl(h^{-1}(x)\bigr).$$

If we compose with $f\colon U \to D$ and note that $h^{-1} = g \circ f^{-1}$, we find that

$$a_j \circ f = \sum_{k=1}^{n} \left( \frac{\partial h_j}{\partial y_k} \circ g \right) \cdot \bigl(b_k \circ g\bigr)$$

Now if $a\colon U \to \mathbb{R}^n$ and $b\colon U \to \mathbb{R}^n$ represent the same smooth section of the tangent bundle, then we can read off the transition functions by comparing the formula we have just derived with (6.2). This leads to the following conclusion.

**Definition 6.3.** Let $M$ be a (connected) smooth manifold of dimension $n$. Cover $M$ by coordinate charts $f_\alpha\colon U_\alpha \to D_\alpha$, where $D_\alpha \subseteq \mathbb{R}^n$ is an open subset with coordinates $x_\alpha = (x_{\alpha,1}, \dots, x_{\alpha,n})$, and as usual set $h_{\alpha,\beta} = f_\alpha \circ f_\beta^{-1}$. Then the *real tangent bundle* $T_{\mathbb{R}}M$ is the smooth vector bundle of rank $n$ defined by the collection of transition functions

$$g_{\alpha,\beta} = J_{\mathbb{R}}(h_{\alpha,\beta}) \circ h_\beta^{-1}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R}),$$

where $J_{\mathbb{R}}(h_{\alpha,\beta}) = \partial h_{\alpha,\beta}/\partial x_\beta$ is the matrix of partial derivates of $h_{\alpha,\beta}$.

**Holomorphic tangent bundles.** Now let $M$ be a complex manifold, and let $p \in M$ be any point. Again, there is an isomorphism $f\colon U \to D$ between a neighborhood of $p$ and an open subset $D \subseteq \mathbb{C}^n$, satisfying $f(p) = 0$; it defines a local holomorphic coordinate system $z_1, \dots, z_n \in \mathscr{O}_M(U)$ centered at the point $p$.

We can write $z_j = x_j + i y_j$, where both $x_j$ and $y_j$ are smooth real-valued functions on $U$. Then $(x_1, \dots, x_n, y_1, \dots, y_n)$ gives an isomorphism between $U$ and an open subset of $\mathbb{R}^{2n}$; this illustrates the obvious fact that $M$ is also a smooth manifold of real dimension $2n$. Consequently, the *real tangent space* at the point $p$ is now

$$T_{\mathbb{R},p}M = \mathbb{R}\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\}.$$

Another useful notion is the *complexified tangent space*

$$\begin{aligned} T_{\mathbb{C},p}M &= \mathbb{C}\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\} \\ &= \mathbb{C}\left\{ \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\}, \end{aligned}$$

where the alternative basis in the second line is again given by

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Finally, the two subspaces

$$T'_p M = \mathbb{C}\left\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right\} \quad \text{and} \quad T''_p M = \mathbb{C}\left\{\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}\right\}$$

of the complexified tangent space are called the *holomorphic* and *antiholomorphic tangent spaces*, respectively.

The holomorphic and antiholomorphic tangent spaces give a direct sum decomposition

$$T_{\mathbb{C},p}M = T'_p M \oplus T''_p M.$$

Evidently, $\partial/\partial\bar{z}_j$ is the complex conjugate of $\partial/\partial z_j$, and so complex conjugation interchanges $T'_p M$ and $T''_p M$. Therefore the map

$$T_{\mathbb{R},p}M \hookrightarrow T_{\mathbb{C},p}M \twoheadrightarrow T'_p M$$

is an isomorphism of $\mathbb{R}$-vector spaces; it maps $\partial/\partial x_j$ to $\partial/\partial z_j$ and $\partial/\partial y_j$ to $i \cdot \partial/\partial z_j$. The relationship between the different tangent spaces is one of the useful features of calculus on complex manifolds.

*Example* 6.4. The holomorphic tangent spaces $T'_p M$ are the fibers of a holomorphic vector bundle $T'M$, the *holomorphic tangent bundle* of $M$.

To describe a set of transition functions for the tanget bundle, we continue to assume that $\dim M = n$, and cover $M$ by coordinate charts $f_\alpha \colon U_\alpha \to D_\alpha$, with $D_\alpha \subseteq \mathbb{C}^n$ open. Let

$$h_{\alpha,\beta} = f_\alpha \circ f_\beta^{-1} \colon f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$$

give the transitions between the charts. Then the differential $J(h_{\alpha,\beta})$ can be viewed as a holomorphic mapping from $f_\beta(U_\alpha \cap U_\beta)$ into $\mathrm{GL}_n(\mathbb{C})$; by analogy with the smooth case, we expect the transition functions for $T'M$ to be given by the formula

$$g_{\alpha,\beta} = J(h_{\alpha,\beta}) \circ f_\beta,$$

where $J(h_{\alpha,\beta}) = \partial h_{\alpha,\beta}/\partial z_\beta$ is now the matrix of all holomorphic partial derivatives. Let us verify that the compatibility conditions in (5.5) hold. By the chain rule,

$$\begin{aligned} g_{\alpha,\beta} \cdot g_{\beta,\gamma} &= \big(J(h_{\alpha,\beta}) \circ f_\beta\big) \cdot \big(J(h_{\beta,\gamma}) \circ f_\gamma\big) = \Big(\big(J(h_{\alpha,\beta}) \circ h_{\beta,\gamma}\big) \cdot J(h_{\beta,\gamma})\Big) \circ f_\gamma \\ &= J(h_{\alpha,\beta} \circ h_{\beta,\gamma}) \circ f_\gamma = J(h_{\alpha,\gamma}) \circ f_\gamma = g_{\alpha,\gamma}, \end{aligned}$$

and so the $g_{\alpha,\beta}$ are the transition functions for a holomorphic vector bundle $\pi \colon T'M \to M$ of rank $n$. The same calculation as in the smooth case shows that sections of $T'M$ are holomorphic vector fields.

**Complex submanifolds.** Let $(X, \mathscr{O}_X)$ be a geometric space, and $Z \subseteq X$ any subset. There is a natural way to make $Z$ into a geometric space: First, we give $Z$ the induced topology. We call a continuous function $f \colon V \to \mathbb{C}$ on an open subset $V \subseteq Z$ *distinguished* if every point $a \in Z$ admits an open neighborhood $U_a$ in $X$, such that there exists $f_a \in \mathscr{O}_X(U_a)$ with the property that $f(z) = f_a(z)$ for every $z \in V \cap U_a$. One can easily check that this defines a geometric structure on $Z$, which we denote by $\mathscr{O}_X|_Z$.

Now suppose that $X$ is a complex manifold. We are interested in finding conditions under which $(Z, \mathscr{O}_X|_Z)$ is also a complex manifold. The following example illustrates the situation.

*Example* 6.5. Consider $\mathbb{C}^k$ as a subset of $\mathbb{C}^n$ (for $n \geq k$), by means of the embedding $(z_1, \dots, z_k) \mapsto (z_1, \dots, z_k, 0, \dots, 0)$. If $f$ is a holomorphic function on an open subset $V \subseteq \mathbb{C}^k$, then $f$ is distinguished in the above sense, since it obviously extends to a holomorphic function on $V \times \mathbb{C}^{n-k}$. Thus we have $\mathscr{O}_{\mathbb{C}^n}|_{\mathbb{C}^k} = \mathscr{O}_{\mathbb{C}^k}$.

The example motivates the following definition.

**Definition 6.6.** A subset $Z$ of a complex manifold $(X, \mathscr{O}_X)$ is called *smooth* if, for every point $a \in Z$, there exists a chart $\phi\colon U \to D \subseteq \mathbb{C}^n$ such that $\phi(U \cap Z)$ is the intersection of $D$ with a linear subspace of $\mathbb{C}^n$. In that case, we say that $(Z, \mathscr{O}_X|_Z)$ is a *complex submanifold* of $X$.

Calling $Z$ a complex submanifold is justified, because $Z$ is obviously itself a complex manifold. Indeed, if $\phi\colon U \to D$ is a local chart for $X$ as in the definition, then the restriction of $\phi$ to $U \cap Z$ provides a local chart for $Z$.