Class 4. Complex manifolds (September 10)

Last time, we defined geometric spaces, which consists of a topological space X (Hausdorff and second countable) together with a distinguished class of functions $\mathscr{O}$. Here is another example:

Example 4.1. Let X be an open set in $\mathbb{R}^n$, and for every open subset $U \subseteq X$, let $\mathscr{A}(U) \subseteq C(U)$ be the subring of smooth (meaning, infinitely differentiable) functions on U. Then $(X, \mathscr{A})$ is again a geometric space.

Definition 4.2. A *morphism* $f\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of geometric spaces is a continuous map $f\colon X \to Y$, with the following additional property: whenever $U \subseteq Y$ is open, and $g \in \mathscr{O}_Y(U)$, the composition $g \circ f$ belongs to $\mathscr{O}_X\big(f^{-1}(U)\big)$.

Example 4.3. Let $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$ be open subsets. We view $(D, \mathscr{O})$ as a geometric space, with the distinguished functions being the holomorphic functions; likewise for E. Then a morphism of geometric spaces $f\colon (D, \mathscr{O}) \to (E, \mathscr{O})$ is the same as a holomorphic mapping $f\colon D \to E$. This is because a continuous map $f\colon D \to E$ is holomorphic iff it preserves holomorphic functions (by Lemma 1.7).

For a morphism $f\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$, we typically write

$$f^*\colon \mathscr{O}_Y(U) \to \mathscr{O}_X\big(f^{-1}(U)\big)$$

for the induced ring homomorphisms. We say that f is an *isomorphism* if it has an inverse that is also a morphism; this means that $f\colon X \to Y$ should be a homeomorphism (of topological spaces), and that each map $f^*\colon \mathscr{O}_Y(U) \to \mathscr{O}_X\big(f^{-1}(U)\big)$ should be an isomorphism of rings.

Example 4.4. If $(X, \mathscr{O})$ is a geometric space, then any open subset $U \subseteq X$ inherits a geometric structure $\mathscr{O}|_U$, by setting $\big(\mathscr{O}|_U\big)(V) = \mathscr{O}(V)$ for $V \subseteq U$ open. With this definition, the natural inclusion map $(U, \mathscr{O}|_U) \to (X, \mathscr{O})$ becomes a morphism.

Complex manifolds. We now define a complex manifold as a geometric space that is locally isomorphic to an open subset of $\mathbb{C}^n$, with the geometric structure given by Example 3.9.

Definition 4.5. A *complex manifold* is a geometric space $(X, \mathscr{O}_X)$ in which every point has an open neighborhood $U \subseteq X$, such that $(U, \mathscr{O}_X|_U) \simeq (D, \mathscr{O})$ for some open subset $D \subseteq \mathbb{C}^n$ and some $n \in \mathbb{N}$.

The integer n is called the *dimension* of the complex manifold X at the point x, and denoted by $\dim_x X$. In fact, it is uniquely determined by the rings $\mathscr{O}_X(U)$, as U ranges over sufficiently small open neighborhoods of x. Namely, define the *local ring* of X at the point x to be

$$\mathscr{O}_{X,x} = \varinjlim_{U \ni x} \mathscr{O}_X(U);$$

as in the case of $\mathscr{O}_n$, its elements are germs of holomorphic functions in a neighborhood of $x \in X$. A moment's thought shows that we have $\mathscr{O}_{X,x} \simeq \mathscr{O}_n$, and therefore $\mathscr{O}_{X,x}$ is a local ring by Theorem 3.1. The integer n can now be recovered from $\mathscr{O}_{X,x}$ by Lemma 2.2, since $n = \dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the ideal of functions vanishing at the point x. In particular, the dimension is preserved under isomorphisms of complex manifolds, and is therefore a well-defined notion.

It follows that the function $x \mapsto \dim_x X$ is locally constant; if X is connected, the dimension is the same at each point, and the common value is called the *dimension* of the complex manifold X, denoted by $\dim X$. In general, the various connected components of X need not be of the same dimension, however.

A morphism of complex manifolds is also called a *holomorphic mapping*; an isomorphism is said to be a *biholomorphic mapping* or a *biholomorphism.* Example 4.3 shows that this agrees with our previous definitions for open subsets of $\mathbb{C}^n$.

Charts and atlases. Note that smooth manifolds can be defined in a similar way: as those geometric spaces that are locally isomorphic to open subsets of $\mathbb{R}^n$ (as in Example 4.1). More commonly, though, smooth manifolds are described by atlases: a collection of charts (or local models) is given, together with transition functions that describe how to pass from one chart to another. Since it is also convenient, let us show how to do the same for complex manifolds.

In the alternative definition, let X be a topological space (again, Hausdorff and with a countable basis). An *atlas* is a covering of X by open subsets $U_i \subseteq X$, indexed by $i \in I$, together with a set of homeomorphisms $\phi_i \colon U_i \to D_i$, where D_i is an open subset of some $\mathbb{C}^n$; the requirement is that the *transition functions*

$$g_{i,j} = \phi_i \circ \phi_j^{-1} \colon \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j),$$

which are homeomorphisms, should actually be biholomorphic mappings. Each $\phi_i \colon U_i \to D_i$ is then called a *coordinate chart* for X, and X is considered to be described by the atlas.

Proposition 4.6. *The alternative definition of complex manifolds is equivalent to Definition 4.5.*

Proof. One direction is straightforward: If we are given a complex manifold $(X, \mathscr{O}_X)$ in the sense of Definition 4.5, we can certainly find for each $x \in X$ an open neighborhood U_x, together with an isomorphism of geometric spaces $\phi_x \colon (U_x, \mathscr{O}_X|_{U_x}) \to (D_x, \mathscr{O})$, for $D_x \subseteq \mathbb{C}^n$ open. Then $g_{x,y}$ is an isomorphism between $\phi_x(D_x \cap D_y)$ and $\phi_y(D_x \cap D_y)$ as geometric spaces, and therefore a biholomorphic map.

For the converse, we assume that the topological space X is given, together with an atlas of coordinate charts $\phi_i \colon U_i \to D_i$. To show that X is a complex manifold, we first have to define a geometric structure: for $U \subseteq X$ open, set

$$\mathscr{O}_X(U) = \left\{ f \in C(U) \;\middle|\; (f|_{U \cap U_i}) \circ \phi_i^{-1} \text{ holomorphic on } \phi_i(U \cap U_i) \text{ for all } i \in I \right\}.$$

The definition makes sense because the transition functions $g_{i,j}$ are biholomorphic. It is easy to see that $\mathscr{O}_X$ satisfies all three conditions in Definition 3.8, and so $(X, \mathscr{O}_X)$ is a geometric space. It is also a complex manifold, because every point has an open neighborhood (namely one of the U_i) that is isomorphic to an open subset of $\mathbb{C}^n$. □

The following class of examples should be familiar to you already.

Example 4.7. Any Riemann surface is a one-dimensional complex manifold; this follows from Proposition 4.6. In fact, Riemann surfaces are precisely the (connected) one-dimensional complex manifolds.

Projective space. Projective space $\mathbb{P}^n$ is the most important example of a compact complex manifold, and so we spend some time defining it carefully. Basically, $\mathbb{P}^n$ is the set of lines in $\mathbb{C}^{n+1}$ passing through the origin. Each such line is spanned by a nonzero vector $(a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$, and two vectors a, b span the same line iff $a = \lambda b$ for some $\lambda \in \mathbb{C}^*$. We can therefore define

$$\mathbb{P}^n = \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{C}^*,$$

and make it into a topological space with quotient topology. Consequently, a subset $U \subseteq \mathbb{P}^n$ is open iff its preimage $q^{-1}(U)$ under the quotient map $q\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is open. It is not hard to see that $\mathbb{P}^n$ is Hausdorff and compact, and that q is an open mapping.

The equivalence class of a vector $a \in \mathbb{C}^{n+1} - \{0\}$ is denoted by $[a]$; thus points of $\mathbb{P}^n$ can be described through their *homogeneous coordinates* $[a_0, a_1, \dots, a_n]$.

We would like to make $\mathbb{P}^n$ into a complex manifold, in such a way that the quotient map q is holomorphic. This means that if f is holomorphic on $U \subseteq \mathbb{P}^n$, then $g = f \circ q$ should be holomorphic on $q^{-1}(U)$, and invariant under scaling the coordinates. We therefore define

$$\begin{aligned}\mathscr{O}_{\mathbb{P}^n}(U) = \big\{\, f \in C(U) \;\big|\; & g = f \circ q \text{ is holomorphic on } q^{-1}(U), \text{ and} \\ & g(\lambda a) = g(a) \text{ for } a \in \mathbb{C}^{n+1} \setminus \{0\} \text{ and } \lambda \in \mathbb{C}^* \,\big\}.\end{aligned}$$

This definition is clearly local, and satisfies the conditions in Definition 3.8. It remains to show that the geometric space $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n})$ is actually a complex manifold. For this, we note that $\mathbb{P}^n$ is covered by the open subsets

$$U_i = \big\{\, [a] \in \mathbb{P}^n \;\big|\; a_i \neq 0 \,\big\}.$$

To simplify the notation, we consider only the case $i = 0$. The map

$$\phi_0 \colon U_0 \to \mathbb{C}^n, \quad [a] \mapsto \big(a_1/a_0, \dots, a_n/a_0\big)$$

is a homeomorphism; its inverse is given by sending $z \in \mathbb{C}^n$ to the point with homogeneous coordinates $[1, z_1, \dots, z_n]$.

We claim that ϕ_0 is an isomorphism between the geometric spaces $(U_0, \mathscr{O}_{\mathbb{P}^n}|_{U_0})$ and $(\mathbb{C}^n, \mathscr{O})$. Since it is a homeomorphism, we only need to show that ϕ_0 induces an isomorphism between $\mathscr{O}(D)$ and $\mathscr{O}_{\mathbb{P}^n}\big(\phi_0^{-1}(D)\big)$, for any open set $D \subseteq \mathbb{C}^n$. This amounts to the following statement: a function $f \in C(D)$ is holomorphic iff $g = f \circ \phi_0 \circ q$ is holomorphic on $(\phi_0 \circ q)^{-1}(D)$. But that is almost obvious: on the one hand, we have

$$f(z_1, \dots, z_n) = g(1, z_1, \dots, z_n),$$

and so f is holomorphic if g is; on the other hand, on the open set where $a_0 \neq 0$, we have

$$g(a_0, a_1, \dots, a_n) = f\big(a_1/a_0, \dots, a_n/a_n\big),$$

and so g is holomorphic if f is. Similarly, one proves that each U_i is isomorphic to $\mathbb{C}^n$ as a geometric space; since $U_0, U_1, \dots, U_n$ together cover $\mathbb{P}^n$, it follows that $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n})$ is a complex manifold in the sense of Definition 4.5.

Quotients. Another basic way to construct complex manifolds is by dividing a given manifold by a group of automorphisms; a familiar example is the construction of elliptic curves as quotients of $\mathbb{C}$ by lattices.

First, a few definitions. An *automorphism* of a complex manifold X is a biholomorphic self-mapping from X onto itself. The *automorphism group* $\mathrm{Aut}(X)$ is the group of all automorphisms. A subgroup $\Gamma \subseteq \mathrm{Aut}(X)$ is said to be *properly discontinuous* if for any two compact subsets $K_1, K_2 \subseteq X$, the intersection $\gamma(K_1) \cap K_2$ is nonempty for only finitely many $\gamma \in \Gamma$. Finally, Γ is said to be *without fixed points* if $\gamma(x) = x$ for some $x \in X$ implies that $\gamma = \mathrm{id}$.

Example 4.8. Any lattice $\Lambda \subseteq \mathbb{C}$ acts on $\mathbb{C}$ by translation; the action is clearly properly discontinuous and without fixed points.

Define X/Γ as the set of equivalence classes for the action of Γ on X; that is to say, two points $x, y \in X$ are equivalent if $y = \gamma(x)$ for some $\gamma \in \Gamma$. We endow X/Γ with the quotient topology, making the quotient map $q \colon X \to X/\Gamma$ continuous. Note that q is also an open mapping: if $U \subseteq X$ is open, then

$$q^{-1}\big(q(U)\big) = \bigcup_{\gamma \in \Gamma} \gamma(U)$$

is clearly open, proving that $q(U)$ is an open subset of the quotient.

Proposition 4.9. *Let X be a complex manifold, and let $\Gamma \subseteq \mathrm{Aut}(X)$ be a properly discontinuous group of automorphisms of X without fixed points. Then the quotient X/Γ is naturally a complex manifold, and the quotient map $q \colon X \to X/\Gamma$ is holomorphic and locally a biholomorphism.*

Note that in order for q to be holomorphic and locally biholomorphic, the geometric structure on the quotient has to be given by

$$\mathscr{O}_{X/\Gamma}(U) = \left\{\, f \in \mathscr{O}_X\big(q^{-1}(U)\big) \;\middle|\; f \circ \gamma = f \text{ for every } \gamma \in \Gamma \,\right\}.$$

Example 4.10. Let $\Lambda \subseteq \mathbb{C}^n$ be a lattice, that is, a discrete subgroup isomorphic to $\mathbb{Z}^{2n}$. Then Λ acts on $\mathbb{C}^n$ by translations, and the action is again properly discontinuous and without fixed points. Proposition 4.9 shows that the quotient is a complex manifold. As in the case of elliptic curves, one can easily show that $\mathbb{C}^n/\Lambda$ is compact; indeed, if $\lambda_1, \dots, \lambda_{2n}$ are a basis for Λ, then the map

$$[0,1]^{2n} \to \mathbb{C}^n/\Lambda, \quad (x_1, \dots, x_{2n}) \mapsto x_1\lambda_1 + \dots + x_{2n}\lambda_{2n} + \Lambda$$

is surjective. $\mathbb{C}^n/\Lambda$ is called a *complex torus* of dimension n.