Class 3. Analytic sets (September 5)

We now come to another property of the ring $\mathscr{O}_n$ that is of great importance in the local theory. Recall that a (commutative) ring A is called *Noetherian* if every ideal of A can be generated by finitely many elements. An equivalent definition is that every increasing chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ has to stabilize. (To see why, note that the union of all I_k is generated by finitely many elements, which will already be contained in one of the I_k.) Also, A is said to be a *local* ring if it is semi-local and Noetherian.

Theorem 3.1. *The ring $\mathscr{O}_n$ is Noetherian, and therefore a local ring.*

Proof. Again, we argue by induction on $n \geq 0$, the case $n = 0$ being trivial. We may assume that $\mathscr{O}_{n-1}$ is already known to be Noetherian. Let $I \subseteq \mathscr{O}_n$ be a nontrivial ideal, and choose a nonzero element $h \in I$. After a change of coordinates, we may assume that h is regular in z_n; by Theorem 2.8, we can then multiply h by a unit and assume from the outset that h is a Weierstraß polynomial.

For any $f \in I$, Theorem 2.10 shows that $f = qh + r$, where $r \in \mathscr{O}_{n-1}[z_n]$. Set $J = I \cap \mathscr{O}_{n-1}[z_n]$; then we have $r \in J$, and so $I = J + (h)$. According to Hilbert's basis theorem, the polynomial ring $\mathscr{O}_{n-1}[z_n]$ is Noetherian; consequently, the ideal J can be generated by finitely many elements $r_1, \dots, r_m$. It follows that I is also finitely generated (by $r_1, \dots, r_m$ together with h), concluding the proof. $\square$

Analytic sets. Our next topic—and one reason for having proved all those theorems about the structure of the ring $\mathscr{O}_n$—is the study of so-called analytic sets, that is, sets defined by holomorphic equations.

Definition 3.2. Let $D \subseteq \mathbb{C}^n$ be an open set. A subset $Z \subseteq D$ is said to be *analytic* if every point $p \in D$ has an open neighborhood U, such that $Z \cap U$ is the common zero set of a collection of holomorphic functions on U.

Note that we are not assuming that $Z \cap U$ is defined by finitely many equations; but we will soon prove that finitely many equations are enough.

Since holomorphic functions are continuous, an analytic set is automatically closed in D; but we would like to know more about its structure. The problem is trivial for $n = 1$: the zero set of a holomorphic function (or any collection of them) is a set of isolated points. In several variables, the situation is again more complicated.

Example 3.3. The zero set $Z(f)$ of a single holomorphic function $f \in \mathscr{O}(D)$ is called a *complex hypersurface*. In one of the exercises, we have seen that $Z(f)$ has Lebesgue measure zero.

We begin our study of analytic sets by considering their local structure; without loss of generality, we may suppose that $0 \in Z$, and restrict our attention to small neighborhoods of the origin. To begin with, note that Z determines an ideal $I(Z)$ in the ring $\mathscr{O}_n$, namely $I(Z) = \{ f \in \mathscr{O}_n \mid f \text{ vanishes on } Z \}$. Since $I(Z)$ contains the holomorphic functions defining Z, it is clear that Z is the common zero locus of the elements of $I(Z)$. Moreover, it is easy to see that if $Z_1 \subseteq Z_2$, then $I(Z_2) \subseteq I(Z_1)$.

The next observation is that, in some neighborhood of 0, the set Z can actually be defined by *finitely* many holomorphic functions. Indeed, on a suitable neighborhood U of the origin, $Z \cap U$ is the common zero locus of its ideal $I(Z)$; but since $\mathscr{O}_n$ is Noetherian, $I(Z)$ is generated by finitely many elements $f_1, \dots, f_r$, say. After shrinking U, we then have $Z \cap U = Z(f_1) \cap \cdots \cap Z(f_r)$ defined by the vanishing of finitely many holomorphic equations.

We say that an analytic set Z is *reducible* if it can be written as a union of two analytic sets in a nontrivial way; if this is not possible, then Z is called *irreducible*. At least locally, irreducibility is related to the following algebraic condition on the ideal $I(Z)$.

Lemma 3.4. *An analytic set Z is irreducible in some neighborhood of $0 \in \mathbb{C}^n$ iff $I(Z)$ is a prime ideal in the ring $\mathscr{O}_n$.*

Proof. Recall that an ideal I in a ring A is called *prime* if, whenever $a \cdot b \in I$, either $a \in I$ or $b \in I$. One direction is obvious: if we have $fg \in I(Z)$, then $Z \subseteq Z(f) \cup Z(g)$; since Z is irreducible, either $Z \subseteq Z(f)$ or $Z \subseteq Z(g)$, which implies that either $f \in I(Z)$ or $g \in I(Z)$. For the converse, suppose that we have a nontrivial decomposition $Z = Z_1 \cup Z_2$. Since Z_1 is the common zero locus of $I(Z_1)$, we can find a holomorphic function $f_1 \in I(Z_1)$ that does not vanish everywhere on Z_2; similarly, we get $f_2 \in I(Z_2)$ that does not vanish everywhere on Z_1. Then the product $f_1 f_2$ belongs to $I(Z)$, while neither of the factors does, contradicting the fact that $I(Z)$ is a prime ideal. $\square$

A useful property of analytic sets is that they can be locally decomposed into irreducible components; this type of result may be familiar to you from algebraic geometry. Here is an example:

Example 3.5. Consider the holomorphic function $y^2 - x^2 - x^3$ on $\mathbb{C}^2$, with coordinates (x, y). Its zero set is an analytic subset of $\mathbb{C}^2$, and one can check that it is irreducible. In a sufficiently small neighborhood of the origin, the function $\sqrt{1+x}$ is holomorphic, and because

$$y^2 - x^2 - x^3 = \left(y - x\sqrt{1+x}\right) \cdot \left(y + x\sqrt{1+x}\right),$$

the analytic set becomes reducibly in a neighborhood of the origin. There are two irreducible components, defined by the two factors $y \pm x\sqrt{1+x}$.

Proposition 3.6. *Let Z be an analytic set in $D \subseteq \mathbb{C}^n$, with $0 \in Z$. Then in some neighborhood of the origin, there is a decomposition $Z = Z_1 \cup \cdots \cup Z_r$ into irreducible analytic sets Z_j. If we require that there are no inclusions among the Z_j, then the decomposition is unique up to reordering.*

Proof. Suppose that Z could not be written as a finite union of irreducible analytic sets. Then Z has to be reducible, and so $Z = Z_1 \cup Z_2$ in some neighborhood of 0. At least one of the two factors is again reducible, say $Z_1 = Z_{1,1} \cup Z_{1,2}$, on a possibly smaller neighborhood of 0. Continuing in this manner, we obtain a strictly decreasing chain of analytic subsets (on smaller and smaller open neighborhoods)

$$Z \supset Z_1 \supset Z_{1,1} \supset \cdots,$$

and correspondingly, a strictly increasing chain of ideals

$$I(Z) \subset I(Z_1) \subset I(Z_{1,1}) \subset \cdots.$$

But $\mathscr{O}_n$ is Noetherian, and hence such a chain cannot exist. We conclude that $Z = Z_1 \cup \cdots \cup Z_r$, where the Z_j are irreducible in a neighborhood of 0, and where we may clearly assume that there are no inclusions $Z_j \subseteq Z_k$ for $j \neq k$.

To prove the uniqueness, let $Z = Z_1' \cup \dots \cup Z_s'$ is another decomposition without redundant terms. Then

$$Z_j' = (Z_j' \cap Z_1) \cup \dots \cup (Z_j' \cap Z_r),$$

and so by irreducibility, $Z_j' \subseteq Z_k$ for some k. Conversely, we have $Z_k \subseteq Z_l'$ for some l, and since the decompositions are irredundant, it follows that $j = l$ and $Z_j' = Z_k$. It is then easy to show by induction that $r = s$ and $Z_j' = Z_{\sigma(j)}$ for some permutation σ of $\{1, \dots, r\}$. $\square$

Implicit mapping theorem. To say more about the structure of analytic sets, we need a version of the implicit function theorem (familiar from multi-variable calculus). It gives a sufficient condition, in terms of partial derivatives of the defining equations, for being able to parametrize the points of an analytic set by an open set in $\mathbb{C}^k$.

We note that if $Z \subseteq D$ is defined by holomorphic equations $f_1, \dots, f_m$, we can equivalently say that $Z = f^{-1}(0)$, where $f \colon D \to \mathbb{C}^m$ is the holomorphic mapping with coordinate functions f_j. We take this more convenient point of view in this section. As usual, we denote the coordinates on $\mathbb{C}^n$ by $z_1, \dots, z_n$. If $f \colon D \to \mathbb{C}^m$ is holomorphic, we let

$$J(f) = \frac{\partial(f_1, \dots, f_m)}{\partial(z_1, \dots, z_n)}$$

be the matrix of its partial derivatives; in other words, $J(f)_{j,k} = \partial f_j / \partial z_k$ for $1 \le j \le m$ and $1 \le k \le n$.

In order to state the theorem, we also introduce the following notation: Suppose that $m \le n$, so that there are fewer equations (namely m) than variables (namely n). Let us write the coordinates on $\mathbb{C}^n$ in the form $z = (z', z'')$ with $z' = (z_1, \dots, z_m)$ and $z'' = (z_{m+1}, \dots, z_n)$. Similarly, we let $r = (r', r'')$, so that $\Delta(0; r) = \Delta(0; r') \times \Delta(0; r'') \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$. For a holomorphic mapping $f \colon D \to \mathbb{C}^n$, we then have

$$J(f) = \bigl(J'(f), J''(f)\bigr),$$

where $J'(f) = \partial f / \partial z'$ is an $m \times m$-matrix, and $J''(f) = \partial f / \partial z''$ is an $m \times (n-m)$-matrix.

Theorem 3.7. *Let f be a holomorphic mapping from an open neighborhood of $0 \in \mathbb{C}^n$ into $\mathbb{C}^m$ for some $m \le n$, and suppose that $f(0) = 0$. If the matrix $J'(f)$ is nonsingular at the point 0, then for some polydisk $\Delta(0; r)$, there exists a holomorphic mapping $\phi \colon \Delta(0; r'') \to \Delta(0; r')$ with $\phi(0) = 0$, such that*

$$f(z) = 0 \text{ for some point } z \in \Delta(0; r) \text{ precisely when } z' = \phi(z'').$$

Proof. The proof is by induction on the dimension m. First consider the case $m = 1$, where we have a single holomorphic function $f \in \mathscr{O}_n$ with $f(0) = 0$ and $\partial f / \partial z_1 \neq 0$. This means that f is regular in z_1 of order 1; by Theorem 2.8, we can therefore write

$$f(z) = u(z) \cdot \bigl(z_1 - a(z_2, \dots, z_n)\bigr),$$

where $u \in \mathscr{O}_n$ is a unit, and $a \in \mathfrak{m}_{n-1}$. Consequently, $u(0) \neq 0$ and $a(0) = 0$; on a suitable polydisk around 0, we therefore obtain the assertion with $\phi = a$.

Now consider some dimension $m > 1$, assuming that the theorem has been proved in dimension $m - 1$. After a linear change of coordinates in $\mathbb{C}^m$, we may further assume that $J'(f) = \mathrm{id}_m$ at the point $z = 0$. Then $\partial f_1 / \partial z_1(0) = 1$, and it follows from the case $m = 1$ that there is a polydisk $\Delta(0; r)$ and a holomorphic function

$\phi_1 \colon \Delta(0; r_2, \dots, r_n) \to \Delta(0; r_1)$ with $\phi_1(0) = 0$, such that $f_1(z) = 0$ precisely when $z_1 = \phi_1(z_2, \dots, z_n)$.

Define a holomorphic mapping $g \colon \Delta(0; r_2, \dots, r_n) \to \mathbb{C}^{m-1}$ by setting

$$g_j(z_2, \dots, z_n) = f_j\big(\phi_1(z_2, \dots, z_n), z_2, \dots, z_n\big)$$

for $2 \leq j \leq m$. Then clearly $g(0) = 0$, and $\partial(g_2, \dots, g_m)/\partial(z_2, \dots, z_m) = \mathrm{id}_{m-1}$ at the point $z = 0$. It follows from the induction hypothesis that, after further shrinking the polydisk $\Delta(0; r)$ if necessary, there is a holomorphic mapping

$$\psi \colon \Delta(0; r'') \to \Delta(0; r_2, \dots, r_m)$$

with $\psi(0) = 0$, such that $g(z_2, \dots, z_n) = 0$ exactly when $(z_2, \dots, z_m) = \psi(z'')$.

Now evidently $f(z) = 0$ at some point $z \in \Delta(0; r)$ iff $z_1 = \phi_1(z_2, \dots, z_n)$ and $g(z_2, \dots, z_n) = 0$. Hence it is clear that the mapping

$$\phi(z) = \Big(\phi_1\big(\psi(z''), z''\big), \psi(z'')\Big)$$

has all the desired properties. □

In fact, as long as the Jacobian matrix $J(f)$ has rank equal to m at the point $0 \in \mathbb{C}^n$, the analytic set $f^{-1}(0)$ can be parametrized, in some neighborhood of the origin, by an open set in $\mathbb{C}^{n-m}$. This follows from the version above after some easy reindexing.

Geometric spaces. The implicit mapping theorem basically means the following: if $J'(f)$ has maximal rank, then Z looks like $\mathbb{C}^{n-m}$ in some neighborhood of the origin. This is one of the basic examples of a complex manifold.

A smooth manifold is a space that locally looks like an open set in $\mathbb{R}^n$; similarly, a complex manifold should be locally like an open set in $\mathbb{C}^n$. To see that something more is needed, take the example of $\mathbb{C}^n$. It is at the same time a topological space, a smooth manifold (isomorphic to $\mathbb{R}^{2n}$), and presumably a complex manifold; what distinguishes between these different structures is the class of functions that one is interested in. In other words, $\mathbb{C}^n$ becomes a smooth manifold by having the notion of smooth function; and a complex manifold by having the notion of holomorphic function.

We now introduce a convenient framework that includes smooth manifolds, complex manifolds, and many other kinds of spaces. Let X be a topological space; we shall always assume that X is Hausdorff and has a countable basis. For every open subset $U \subseteq X$, let $C(U)$ denote the ring of complex-valued continuous functions on U; the ring operations are defined pointwise.

Definition 3.8. A *geometric structure* $\mathscr{O}$ on the topological space X is a collection of subrings $\mathscr{O}(U) \subseteq C(U)$, where U runs over the open sets in X, subject to the following three conditions:

(1) The constant functions are in $\mathscr{O}(U)$.
(2) If $f \in \mathscr{O}(U)$ and $V \subseteq U$, then $f|_V \in \mathscr{O}(V)$.
(3) If $f_i \in \mathscr{O}(U_i)$ is a collection of functions satisfying $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a unique $f \in \mathscr{O}(U)$ such that $f_i = f|_{U_i}$, where $U = \bigcup_{i \in I} U_i$.

The pair $(X, \mathscr{O})$ is called a *geometric space*; functions in $\mathscr{O}(U)$ will sometimes be called *distinguished*.

The second and third condition together mean that being distinguished is a local property; the typical example is differentiability (existence of a limit) or holomorphicity (power series expansion). In the language of sheaves, which will be introduced later in the course, we may summarize them by saying that $\mathscr{O}$ is a subsheaf of the sheaf of continuous functions on X.

Example 3.9. Let D be an open set in $\mathbb{C}^n$, and for every open subset $U \subseteq D$, let $\mathscr{O}(U) \subseteq C(U)$ be the subring of holomorphic functions on U. Since Definition 1.1 is clearly local, the pair $(D, \mathscr{O})$ is a geometric space.

Example 3.10. The pair (X, C) itself is also a geometric space, where every continuous function is distinguished. Obviously, there is no additional information beyond the topological space itself.