Class 25. Kähler manifolds and projective manifolds

At this point, a few words about the nature of projective manifolds are probably in order. Most compact Kähler manifolds are not projective, and the subset of those that are is quite small. To see why this should be, let us consider the space $H^{1,1}_{\mathbb{R}}$, the intersection of $H^{1,1}$ and $H^2(M, \mathbb{R})$ inside $H^2(M, \mathbb{C})$. It consists of those real cohomology classes that can be represented by a closed form of type $(1,1)$. We say that a class $\alpha \in H^{1,1}_{\mathbb{R}}$ is a *Kähler class* if it can be represented by a closed positive $(1,1)$-form. The set of all such forms is a cone (since it is closed under addition, and under multiplication by positive real numbers), the so-called *Kähler cone* of the manifold M. Since positivity is an open condition (by the argument we used in the proof of Corollary 24.3), the Kähler cone is an open subset of $H^{1,1}_{\mathbb{R}}$. Now in order for M to be projective, the Kähler cone has to contain at least one nonzero rational class. But the space $H^2(M, \mathbb{Q}) \subseteq H^2(M, \mathbb{R})$ of all rational classes is a countable subset, and in general, it is unlikely that the Kähler cone will intersect it nontrivially.

Example 25.1. Consider again the case of K3-surfaces, that is, compact Kähler surfaces whose Hodge diamond looks like

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & 0 & & 0 & \\ \mathbb{C} & & \mathbb{C}^{20} & & \mathbb{C} \\ & 0 & & 0 & \\ & & \mathbb{C} & & \end{array}$$

(The precise definition is that a K3 surface is a simply connected compact Kähler surface whose canonical line bundle is trivial.) When discussing Griffith's theorem, we saw that nonsingular quartic hypersurfaces in $\mathbb{P}^3$ are K3-surfaces. The space of homogeneous polynomials of degree 4 has dimension $\binom{4+3}{3} = 35$, and so nonsingular quartic hypersurfaces are naturally parametrized by an open subset in $\mathbb{P}^{34}$. On the other hand, the automorphism group of $\mathbb{P}^3$ has dimension 15, and if we take its action into account, we find that this particular class of K3-surfaces forms a 19-dimensional family.

In the theory of deformations of complex manifolds, it is shown that there is a 20-dimensional manifold P that parametrizes all possible K3-surfaces (20 being the dimension of $H^{1,1}$). Now what about projective K3-surfaces? They form a dense subset of P, consisting of countably many analytic subsets of dimension 19. So, just as in the case of those K3-surfaces that can be realized as quartic surfaces in $\mathbb{P}^3$, projective K3-surfaces always come in 19-dimensional families; but altogether, they are still a relatively sparse subset of the space of all K3-surfaces.

Why are the subsets corresponding to projective K3-surfaces all of dimension 19? The answer has to do with the Hodge decomposition on $H^2(M, \mathbb{C})$. Let us fix some projective K3-surface M_0, and consider those M that are close to M_0 on the moduli space P. It is possible to identify the cohomology group $H^2(M, \mathbb{Z})$ with $H^2(M_0, \mathbb{Z})$, and hence $H^2(M, \mathbb{C})$ with $H^2(M_0, \mathbb{C})$. We can then think of the Hodge decomposition on $H^2(M, \mathbb{C})$ as giving us a decomposition of the fixed 22-dimensional vector space $H^2(M_0, \mathbb{C})$ into subspaces of dimension 1, 20, and 1. (This is an example of a so-called variation of Hodge structure.)

M_0 being projective, there exists $\omega_0 \in H^2(M_0, \mathbb{Z})$ whose class in $H^2(M_0, \mathbb{C})$ is represented by a closed positive $(1,1)$-form. Through the isomorphism $H^2(M, \mathbb{Z}) \simeq H^2(M_0, \mathbb{Z})$, we get a class $\omega_M \in H^2(M, \mathbb{Z})$ on every nearby K3-surface M. If M is to remain projective, then this class should still be of type $(1,1)$, which means that its image in $H^{0,2}(M)$ should be zero. Since $\dim H^{0,2}(M) = 1$, this is one condition, and so the set of M where $\omega_M \in H^{1,1}(M)$ will be a hypersurface in P (positivity is automatic if M is close to M_0).

A complex torus without geometry. To illustrate how far a general compact Kähler manifold is from being projective, we shall now look at an example of a 2-dimensional complex torus T in which the only analytic subsets are points and T itself. In contrast to this, a submanifold of projective space always has a very rich geometry, since there are many analytic subsets obtained by intersecting with various linear subspaces of projective space. The torus T in the example (due to Steven Zucker) can therefore not be embedded into projective space.

Let $V = \mathbb{C} \oplus \mathbb{C}$, with coordinates (z, w), and let $J \colon V \to V$ be the complex-linear mapping defined by $J(z, w) = (iz, -iw)$. Let $\Lambda \subseteq V$ be a lattice with the property that $J(\Lambda) = \Lambda$, and form the 2-dimensional complex torus $T = V/\Lambda$. Then J induces an automorphism of T, and we refer to T as a *J-torus*. Any lattice of this type can be described by a basis of the form v_1, v_2, Jv_1, Jv_2, and is thus given by a 2×4-matrix

$$\begin{pmatrix} a & b & ia & ib \\ c & d & -ic & -id \end{pmatrix}$$

with complex entries. Here $a, b, c, d \in \mathbb{C}$ need to be chosen such that the four column vectors of the matrix are linearly independent over $\mathbb{R}$, but are otherwise arbitrary. In this way, we have a whole four-dimensional family of J-tori. We shall assume in addition that $a\bar{d} - b\bar{c} \neq 0$.

Lemma 25.2. *If we let $f = a\bar{d} - b\bar{c}$, then both the real and the imaginary part of $\theta = f^{-1} dz \wedge d\bar{w}$ are closed $(1,1)$-forms with integral cohomology class.*

Proof. Both the real and the imaginary part of θ are closed forms of type $(1,1)$, because $\operatorname{Re}\theta = \frac{1}{2}(\theta + \overline{\theta})$ and $\operatorname{Im}\theta = \frac{1}{2i}(\theta - \overline{\theta})$. As explained before, we have $\Lambda = H_1(T, \mathbb{Z})$, and so to show that a closed form defines an integral cohomology class, it suffices to evaluate it on vectors in Λ. If we substitute (u_1, v_1) and (u_2, v_2) into the form $dz \wedge d\bar{w}$, we obtain $u_1\overline{v_2} - u_2\overline{v_1}$. The 16 evaluations of $dz \wedge d\bar{w}$ can thus be summarized by the matrix computation

$$\begin{pmatrix} -\bar{c} & a \\ -\bar{d} & b \\ -i\bar{c} & ia \\ -i\bar{d} & ib \end{pmatrix} \begin{pmatrix} a & b & ia & ib \\ \bar{c} & \bar{d} & i\bar{c} & i\bar{d} \end{pmatrix} = \begin{pmatrix} 0 & a\bar{d} - b\bar{c} & 0 & i(a\bar{d} - b\bar{c}) \\ b\bar{c} - a\bar{d} & 0 & i(b\bar{c} - a\bar{d}) & 0 \\ 0 & i(a\bar{d} - b\bar{c}) & 0 & b\bar{c} - a\bar{d} \\ i(b\bar{c} - a\bar{d}) & 0 & a\bar{d} - b\bar{c} & 0 \end{pmatrix},$$

which proves that all values of θ on $\Lambda \times \Lambda$ are contained in the set $\{0, \pm 1, \pm i\}$. $\square$

Now let $\alpha = \operatorname{Re}\theta$ and $\beta = \operatorname{Im}\theta$; both are closed $(1,1)$-forms with integral cohomology class. Our next goal is to show that, for a generic lattice Λ (corresponding to a generic choice of $a, b, c, d \in \mathbb{C}$), these are the only cohomology classes that are both integral and of type $(1,1)$.

Lemma 25.3. *If the lattice Λ is generic, then $H^2(T, \mathbb{Z}) \cap H^{1,1}(T) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$.*

Proof. Let e_1, e_2, e_3, e_4 be the four basis vectors of Λ, and let $e_1^*, e_2^*, e_3^*, e_4^* \in H^1(T, \mathbb{Z})$ be the dual basis. According to the calculation above, we then have

$$\alpha = e_1^* \wedge e_2^* - e_3^* \wedge e_4^* \qquad \text{and} \qquad \beta = e_1^* \wedge e_4^* - e_2^* \wedge e_3^*.$$

We can now write any element in $H^2(T, \mathbb{Z})$ in the form

$$\varphi = \sum_{1 \le j < k \le 4} u_{j,k} e_j^* \wedge e_k^*,$$

where the six coefficients $u_{j,k}$ are integers. In order for this form to be of type $(1,1)$, what has to happen is that $dz \wedge dw \wedge \varphi = 0$. For every choice of integers $u_{j,k}$, this is a polynomial equation in the four complex numbers a, b, c, d.

What are those equations? By a computation similar to the above, one has

$$\begin{pmatrix} -c & a \\ -d & b \\ ic & ia \\ id & ib \end{pmatrix} \begin{pmatrix} a & b & ia & ib \\ c & d & -ic & -id \end{pmatrix} = \begin{pmatrix} 0 & ad - bc & -2iac & -i(ad+bc) \\ bc - ad & 0 & -i(ad+bc) & -2ibd \\ 2iac & i(ad+bc) & 0 & ad - bc \\ i(ad+bc) & 2ibd & bc - ad & 0 \end{pmatrix},$$

from which it follows that

$$dz \wedge dw = (ad - bc)(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*) - i(ad+bc)(e_1^* \wedge e_4^* + e_2^* \wedge e_3^*) - 2iac e_1^* \wedge e_3^* - 2ibd e_2^* \wedge e_4^*.$$

After simplifying the resulting formulas, we find that $dz \wedge dw \wedge \varphi = C e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*$, where the coefficient is given by

$$C = (ad - bc)(u_{3,4} + u_{1,2}) - i(ad+bc)(u_{2,3} + u_{1,4}) + 2iac u_{2,4} + 2ibd u_{1,3}.$$

To complete the proof, we have to show that for a general choice of $(a, b, c, d) \in \mathbb{C}^4$, the equation $C = 0$ can only be satisfied if φ is a linear combination of α and β.

By subtracting suitable multiples of α and β, we may assume that $u_{3,4} = u_{2,3} = 0$. We are then left with the equation

$$(ad - bc)u_{1,2} - i(ad+bc)u_{1,4} + 2iac u_{2,4} + 2ibd u_{1,3} = 0.$$

If we now set $a = xb$ and $c = yd$, and choose $x, y \in \mathbb{C}$ algebraically independent over $\mathbb{Q}$, we arrive at

$$(x - y)u_{1,2} - i(x+y)u_{1,4} + 2ixy u_{2,4} + 2i u_{1,3},$$

which clearly has no nontrivial solution in integers $u_{1,2}, u_{1,4}, u_{2,4}, u_{1,3}$. This proves that each of the polynomial equations above defines a proper analytic subset of $\mathbb{C}^4$, and consequently of measure zero. We have countably many of these sets (parametrized by the choice of $u_{j,k}$), and it follows that the set of parameters $(a, b, c, d) \in \mathbb{C}^4$ for which the corresponding J-torus satisfies $H^{1,1}(T) \cap H^2(T, \mathbb{Z}) \neq \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ has measure zero. $\square$

From now on, we let T be a generic J-torus in the sense of Lemma 25.3. Recall that J defines an automorphism of T. It is easy to see that we have $J^*\theta = f^{-1}(idz) \wedge (id\bar{w}) = -\theta$, and hence $J^*\alpha = -\alpha$ and $J^*\beta = -\beta$. Since T is generic, we conclude that $J^*\varphi = -\varphi$ for every class $\varphi \in H^2(T, \mathbb{Z}) \cap H^{1,1}(T)$.

Lemma 25.4. *If T is a generic J-torus, then T contains no analytic subsets of dimension one.*

Proof. We will first show that T contains no one-dimensional complex submanifolds. Suppose to the contrary that $C \subseteq T$ was such a submanifold. Integration over C defines a cohomology class $[C] \in H^2(T,\mathbb{Z}) \cap H^{1,1}(T)$, and by the calculation above, we have $[J^{-1}C] = J^*[C] = -[C]$. This shows that $[C] + [J^{-1}C] = 0$. But such an identity is impossible on a compact Kähler manifold: letting ω be the Kähler form of the natural Kähler metric on T, the integral

$$\int_T \omega \wedge \big([C] + [J^{-1}C]\big) = \int_C \omega|_C + \int_{J^{-1}C} \omega|_{J^{-1}C} = \mathrm{vol}(C) + \mathrm{vol}(J^{-1}C)$$

is the volume of the two submanifolds with respect to the induced metric, and hence positive. This is a contradiction, and so it follows that T cannot contain any one-dimensional submanifolds.

Similarly, if $Z \subseteq T$ is a one-dimensional analytic subset, one can show that integration over the set of smooth points of Z (the complement of a finite set of points) defines a cohomology class $[Z] \in H^2(T,\mathbb{Z}) \cap H^{1,1}(T)$, whose integral against the Kähler form ω is positive. As before, we conclude that there cannot be such analytic subsets in a generic J-torus T. ☐

The Levi extension theorem. To conclude our discussion of the class of compact Kähler manifolds that can be embedded into projective space, we will prove Chow's theorem: every complex submanifold of $\mathbb{P}^n$ is defined by polynomial equations, and hence an algebraic variety. We will deduce this from an extension theorem for analytic sets, known as the *Levi extension theorem*. First, recall a basic definition from earlier in the semester: a closed subset Z of a complex manifold M is said to be analytic if, for every point $p \in Z$, there are locally defined holomorphic functions $f_1, \dots, f_r \in \mathscr{O}_M(U)$ such that $Z \cap U = Z(f_1, \dots, f_r)$ is the common zero set.

Here is the statement of the extension theorem (first proved in this form by the two German mathematicians Remmert and Stein).

Theorem 25.5. *Let M be a connected complex manifold of dimension n, and let $Z \subseteq M$ be an analytic subset of codimension at least $k+1$. If $V \subseteq M \setminus Z$ is an analytic subset of codimension k, then the closure $\overline{V}$ in M remains analytic.*

Example 25.6. Recall the following special case of Hartog's theorem: if f is a holomorphic function on $M \setminus \{p\}$, and if $\dim M \geq 2$, then f extends to a holomorphic function on M. In the same situation, Levi's theorem shows that if $V \subseteq M \setminus \{p\}$ is an analytic subset of codimension 1, then its closure $\overline{V}$ is analytic in M. The Levi extension theorem may thus be seen as a generalization of Hartog's theorem from holomorphic functions to analytic sets.

We begin the proof by making several reductions. In the first place, it suffices to prove the statement under the additional assumption that $Z \subseteq M$ is a submanifold of codimension $\geq k$. The general case follows from this by the following observation: by one of the exercises, the set of singular points of Z (i.e., those points where Z is not a submanifold of M) is contained in a proper analytic subset Z_1. Similarly, the set of singular points of Z_1 is contained in a proper analytic subseteq $Z_2 \subset Z_1$. Thus we have a chain $Z = Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ of closed analytic sets, with each $Z_j \setminus Z_{j+1}$ a complex submanifold of codimension $\geq k$ in M. Since there can be no infinite strictly decreasing chains of analytic sets, we have $Z_{r+1} = \emptyset$ for some $r \in \mathbb{N}$. We may now extend V successively over the submanifolds $Z_j \setminus Z_{j+1}$, by first taking the closure of V in $M \setminus Z_1$, then in $M \setminus Z_2$, and so on.

In the second place, the definition of analytic sets is local, and so we only need to show that $\overline{V}$ is analytic in a neighborhood of any of its points. We may therefore assume in addition that M is a polydisk in $\mathbb{C}^n$ containing the origin, and that $0 \in Z$. After a suitable change of coordinates, we can furthermore arrange that the submanifold Z is of the form $z_1 = z_2 = \cdots = z_{k+1} = 0$.

Thus the general case of Levi's theorem is reduced to the following local statement.

Proposition 25.7. *Let $D \subseteq \mathbb{C}^n$ be a polydisk containing the origin, and let $Z = Z(z_1, \dots, z_{k+1})$. If V is an analytic subset of $D \setminus Z$ of codimension k, then $\overline{V}$ is an analytic subset of D.*

For simplicity, we shall only give the proof in the case $k = 1$ and $n = 2$. Exactly the same argument works for $k = 1$ and arbitrary n, except that the notation becomes more cumbersome; to prove the general case, one needs to know slightly more about the local structure of analytic sets than we have proved.

To fix the notation, let us say that $D = \Delta^2$ is the set of points $(z, w) \in \mathbb{C}^2$ with $|z| < 1$ and $|w| < 1$, and that Z consists of the point $(0, 0)$. Furthermore, V is an analytic subset of $D \setminus \{(0, 0)\}$ of dimension one, and we may clearly choose the coordinate system in such a way that the line $z = 0$ is not contained in V. We will prove the theorem by explicitly constructing a holomorphic function $H \in \mathscr{O}(D)$ whose zero locus is $\overline{V}$.

Let $D' = \Delta^* \times \Delta$ be the set of points in D where $z \neq 0$. We first want to show that $V' = V \cap D'$ is defined by the vanishing of a single holomorphic function on D'. Consider the associated line bundle $\mathscr{O}_{D'}(-V')$. We already know that $H^1(D', \mathscr{O}) \simeq 0$ and $H^2(D', \mathbb{Z}) \simeq 0$, and so the long exact sequence coming from the exponential sequence shows that $H^1(D', \mathscr{O}^*) \simeq 0$. We conclude that the line bundle $\mathscr{O}_{D'}(-V')$ is trivial, and hence that there is a holomorphic function $h \in \mathscr{O}(D')$ whose zero set is the divisor V'. The rest of the proof consists in suitably extending h to a holomorphic function H on a neighborhood of the origin in D.

Since V does not contain the line $z = 0$, the intersection $V \cap Z(z)$ consists of a discrete set of points in the punctured disk $0 < |w| < 1$. We may thus find a small circle, say of radius $\varepsilon > 0$, that does not meet any of these points. By continuity, the set of points (z, w) with $|z| \leq \delta$ and $|w| = \varepsilon$ will not meet V, provided that we choose $\delta > 0$ sufficiently small.

Now we claim that V intersects each vertical disk in the same number of points. For fixed z with $0 < |z| \leq \delta$, that number is given by the integral

$$d(z) = \frac{1}{2\pi i} \int_{|w|=\varepsilon} \frac{1}{h(z,w)} \frac{\partial h(z,w)}{\partial w} dw \in \mathbb{Z},$$

which counts the zeros of the holomorphic function $h(z, -)$ inside the disk $|w| < \varepsilon$. Since $d(z)$ is continuous and integer-valued, it has to be constant; let $d = d(0)$ be the constant value.

For fixed z with $0 < |z| \leq \delta$, we let $w_1(z), \dots, w_d(z)$ be the w-coordinates of the intersection points (in any order). The power sums

$$\sum_{j=1}^{d} w_j(z)^k = \frac{1}{2\pi i} \int_{|w|=\varepsilon} \frac{w^k}{h(z,w)} \frac{\partial h(z,w)}{\partial w} dw$$

are evidently holomorphic functions of z as long as $0 < |z| < \delta$. By Newton's identities, the same is therefore true for the elementary symmetric functions $\sigma_k(z)$.

On the other hand, $|\sigma_k(z)|$ is clearly bounded by the quantity $\binom{d}{k} \cdot \varepsilon^k$, and therefore extends to a holomorphic function on the set $|z| < \delta$ by Riemann's theorem.

If we now define

$$H(z, w) = w^d - \sigma_1(z) w^{d-1} + \sigma_2(z) w^{d-2} + \cdots + (-1)^d \sigma_d(z),$$

then H is a holomorphic function for $|z| < \delta$ and $|w| < \varepsilon$, whose roots for fixed $z \neq 0$ are exactly the points $w_1(z), \dots, w_d(z)$. Its zero set $Z(H)$ is a closed analytic set which, by construction, contains all points of V that satisfy $0 < |z| < \delta$ and $|w| < \varepsilon$. It is then not hard to see that $Z(H) = \overline{V}$ on the open subset where $|z| < \delta$ and $|w| < \varepsilon$, proving that $\overline{V}$ is indeed analytic.