Class 24. The Kodaira embedding theorem and applications (November 21)

We are in the middle of proving the Kodaira embedding theorem. We were considering the blowup $\pi\colon \tilde{M} \to M$ at a point $P \in M$, and wanted to show that if L is a positive line bundle on M, then the pullback line bundle $\tilde{L} = \pi^* L$ can be made positive in the following manner.

Lemma. *Let L be a positive line bundle on M. Then for sufficiently large k, the line bundle $\tilde{L}^k \otimes [-E]$ is again positive.*

Proof. Recall that a real $(1,1)$-form α is said to be positive if $\alpha(\xi, \bar{\xi}) > 0$ for every nonzero tangent vector $\xi \in T'_p M$. A holomorphic line bundle is positive if it admits a Hermitian metric for which the real $(1,1)$-form $\frac{i}{2\pi}\Theta$ is positive.

We give the pullback line bundle $\tilde{L} = \pi^* L$ the induced Hermitian metric. Since L is positive, its first Chern class $\omega = \frac{i}{2\pi}\Theta_L$ is a positive form, and so $\frac{i}{2\pi}\Theta_{\tilde{L}} = \pi^*\omega$ is positive outside the exceptional divisor E. At points of E, however, the form $\pi^*\omega$ fails to be positive—more precisely, we have $(\pi^*\omega)(\xi, \bar{\xi}) = 0$ for any ξ that is tangent to E—because the restriction of $\tilde{L}$ to E is trivial. The idea is to construct a Hermitian metric h_E on $[-E]$ which is positive in the directions tangent to E; by choosing $k \gg 0$, we can then make sure that $\Omega_k = \pi^*\omega + \frac{i}{2\pi}\Theta_E$, which represents the first Chern class of $\tilde{L}^k \otimes [-E]$, is a positive form on $\tilde{M}$.

To construct that metric, let U be an open neighborhood of the point p, isomorphic to an open polydisk $D \subseteq \mathbb{C}^n$, and let $z_1, \dots, z_n$ be the resulting holomorphic coordinate system centered at p. Then $U_1 = \pi^{-1}(U)$ is isomorphic to $\mathrm{Bl}_0 D$, the blow-up of the origin in D, which we originally constructed as a submanifold of the product $D \times \mathbb{P}^{n-1}$. We may thus view U_1 itself as being a submanifold of $U \times \mathbb{P}^{n-1}$; under this identification, the line bundle $[-E]$ is isomorphic to the pullback of $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ by the map $q\colon U_1 \to \mathbb{P}^{n-1}$. The latter has a canonical metric, and so we get a Hermitian metric h_1 on the restriction of $[-E]$ to the open set U_1. Note that $i/2\pi$ times its curvature form is equal to the pullback $q^*\omega_{FS}$ of the Fubini-Study from $\mathbb{P}^{n-1}$.

Let $M^* = M \setminus \{p\}$; by construction, the map π is an isomorphism between $U_2 = \tilde{M} \setminus E$ and M^*, and since $[-E]$ is trivial on the complement of E, it has a distinguished nowhere vanishing section s_E over U_2, corresponding to the constant function $1 \in \mathscr{O}_M(M^*)$. We can thus put a Hermitian metric h_2 on the restriction of $[-E]$ to U_2, by declaring the pointwise norm of s_E to be 1. Now let $\rho_1 + \rho_2 = 1$ be a partition of unity subordinate to the open cover U, M^*, and define a Hermitian metric on $[-E]$ by setting

$$h_E = (\rho_1 \circ \pi) h_1 + (\rho_2 \circ \pi) h_2.$$

This is well-defined, and indeed a Hermitian metric (because the convex combination of two Hermitian inner products on a vector space is again a Hermitian inner product).

To complete the proof, we have to argue that $\Omega_k = \frac{i}{2\pi}\Theta_E + k\pi^*\omega$ is a positive form if $k \gg 0$. First consider the open set $U_1 = \pi^{-1}(U)$ containing the exceptional divisor. For any $k > 0$, the form $k \cdot pr_1^*\omega + pr_2^*\omega_{FS}$ on the product $U \times \mathbb{P}^{n-1}$ is clearly positive. In a sufficiently small neighborhood V of the exceptional divisor (namely outside the support of $\rho_2 \circ \pi$), Ω_k is the restriction of that form to the submanifold U_1, and is therefore positive as well. On the other hand, the complement $\tilde{M} \setminus V$ of that neighborhood is a compact set in $\tilde{M} \setminus E$, on which $\frac{i}{2\pi}\Theta_E$ is bounded and $\pi^*\omega$ is positive. By taking k sufficiently large, we can therefore make Ω_k be positive on $\tilde{M} \setminus V$ as well. $\square$

Proof of the embedding theorem. We now come to the proof of Theorem 23.3. We continue to let M be a compact complex manifold, and $L \to M$ a positive line bundle. In order to prove the embedding theorem, we have to show that for $k \gg 0$, the following three things are true:

(1) The line bundle L^k is base-point free, and therefore defines a holomorphic mapping $\varphi_{L^k} \colon M \to \mathbb{P}^{N_k}$, where $N_k = \dim H^0(M, L^k) - 1$. Equivalently, for every point $p \in M$, the restriction map $H^0(M, L^k) \to L_p^k$ is surjective.
(2) The mapping φ_{L^k} is injective; equivalently, for every pair of distinct points $p, q \in M$, the restriction map $H^0(M, L^k) \to L_p^k \oplus L_q^k$ is surjective.
(3) The mapping φ_{L^k} is an immersion, which means that its differential is injective; equivalently, the map $H^0(M, L^k)(p) \to T_p^{1,0} M \otimes L_p^k$ is surjective at every point $p \in M$.

In each of the three cases, the strategy is to blow up the point (or points) in question, and to reduce the surjectivity to the vanishing of some cohomology group on the blow-up. We then show that, after choosing $k \gg 0$, the group is question is zero by Kodaira's theorem.

We shall divide the proof into four steps, which are fairly similar to each other.

Step 1. To show that L^k is base-point free for $k \gg 0$, we begin by proving that for every fixed point $p \in M$, the map $H^0(M, L^k) \to L_p^k$ is surjective once k is large. Let $\pi \colon \tilde{M} \to M$ denote the blow-up of M at the point p, and let $E = \pi^{-1}(p)$ be the exceptional divisor. Let $i \colon E \hookrightarrow \tilde{M}$ be the inclusion map, and let $\tilde{L} = \pi^* L$ be the pullback of the line bundle. Every section of L on M defines by pullback a section of $\tilde{L} = \pi^* L$ on $\tilde{M}$. The resulting linear map

$$H^0(M, L^k) \to H^0(\tilde{M}, \tilde{L}^k)$$

is an isomorphism by Hartog's theorem. Indeed, suppose that $\tilde{s}$ is a global section of $\tilde{L}^k$. Since $\tilde{M} \setminus E \simeq M^*$, the restriction of $\tilde{s}$ to $\tilde{M} \setminus E$ gives a holomorphic section of L^k over M^*. If $n \geq 2$, then Hartog's theorem shows that this section extends holomorphically over the point p, proving that $\tilde{s}$ is in the image of $H^0(M, L^k)$. (If $n = 1$, we have $\tilde{M} = M$ and $E = \{p\}$, and so the statement is trivial.)

Now clearly a section of L^k vanishes at the point p iff the corresponding section of $\tilde{L}^k$ vanishes along the exceptional divisor E; in other words, we have a commutative diagram

$$\begin{tikzcd} H^0(M, L^k) \arrow[r] \arrow[d, "\cong"] & L_p^k \arrow[d, "\cong"] \\ H^0(\tilde{M}, \tilde{L}^k) \arrow[r] & H^0(E, i^* \tilde{L}^k). \end{tikzcd}$$

Note that $i^* \tilde{L}^k \simeq \mathscr{O}_E \otimes L_p^k$, since the restriction of $\tilde{L}^k$ to the exceptional divisor is the trivial line bundle with fiber L_p^k. It is therefore sufficient to prove that, on $\tilde{M}$, the restriction map $H^0(\tilde{M}, \tilde{L}^k) \to H^0(E, i^* \tilde{L}^k)$ is surjective.

Because of the long exact cohomology sequence

$$H^0(\tilde{M}, \tilde{L}^k) \to H^0(E, i^* \tilde{L}^k) \to H^1\big(\tilde{M}, \tilde{L}^k \otimes [-E]\big),$$

the surjectivity is a consequence of $H^1\big(\tilde{M}, \tilde{L}^k \otimes [-E]\big) \simeq 0$. This will follow from the Kodaira vanishing theorem, provided we can show that

$$\tilde{L}^k \otimes [-E] \simeq K_{\tilde{M}} \otimes P_k$$

for some positive line bundle P_k. By Lemma 23.5, we have $K_{\tilde{M}} \simeq \pi^* K_M \otimes [E]^{n-1}$, and so

$$P_k \simeq \pi^*\big(L^k \otimes K_M^{-1}\big) \otimes [-E]^n.$$

Now fix a sufficiently large integer ℓ, with the property that $L^\ell \otimes K_M^{-1}$ is positive. By Lemma 23.6, there exists an integer m_0 such that the line bundle $\tilde{L}^m \otimes [-E]$ is positive for $m \geq m_0$. But then

$$\pi^*\big(L^\ell \otimes K_M^{-1}\big) \otimes \big(\tilde{L}^m \otimes [-E]\big)^n \simeq \pi^*\big(L^{mn+\ell} \otimes K_M^{-1}\big) \otimes [-E]^n$$

is positive, and so it suffices to take $k \geq m_0 n + \ell$. With this choice of k, we have

$$H^1\big(\tilde{M}, \tilde{L}^k \otimes [-E]\big) \simeq H^1\big(\tilde{M}, K_{\tilde{M}} \otimes P_k\big) \simeq 0,$$

which vanishes by Theorem 22.2 because P_k is a positive line bundle. So if $k \geq m_0 n + \ell$, then the restriction map $H^0(M, L^k) \to L_p^k$ is surjective.

Unfortunately, the value of m_0 might depend on the point $p \in M$ that we started from. To show that one value works for all points $p \in M$, we use a compactness argument. Namely, if $H^0(M, L^k) \to L_p^k$ is surjective at some point $p \in M$, it means that L^k has a section that does not vanish at p. The same section is nonzero at nearby points, and so the restriction map is surjective on some neigborhood of the point. We can therefore cover M by open sets U_i, such that the restriction map is surjective for $k \geq k_i$. By compactness, finitely many of these open sets cover M, and if we let k_0 be the maximum of the corresponding k_i, then we get surjectivity at all points for $k \geq k_0$. We have now shown that the mapping φ_{L^k} is well-defined and holomorphic for sufficiently large values of k.

Step 2. Exactly the same proof shows that, given any pair of distinct points $p, q \in M$, the restriction map $H^0(M, L^k) \to L_p^k \oplus L_q^k$ is surjective for $k \gg 0$. We only need to let $\pi\colon \tilde{M} \to M$ be the blow-up of M at both points, and $E = \pi^{-1}(p) \cup \pi^{-1}(q)$ the union of the two exceptional divisors (which is still a submanifold of dimension $n-1$). If $i\colon E \hookrightarrow \tilde{M}$ denotes the inclusion, it suffices to prove the surjectivity of

$$H^0(\tilde{M}, \tilde{L}^k) \to H^0(E, i^* \tilde{L}^k),$$

which holds for the same reason as before once $k \gg 0$. Note that the value of k now depends on the pair of points $p, q \in M$; but this time, we cannot use the same compactness proof because $M \times M \setminus \Delta$ is no longer compact. We will deal with this issue in the last step of the proof.

Step 3. Next, we prove that for a fixed point $p \in M$, the map

$$H^0(M, L^k)(p) \to T_p^{1,0} M \otimes L_p^k$$

becomes surjective if $k \gg 0$. Here $H^0(M, L^k)(p)$ denotes the space of sections of L^k that vanish at the point p. Again let $\pi\colon \tilde{M} \to M$ be the blow-up of M at the point p, let $i\colon E \hookrightarrow \tilde{M}$ be the inclusion of the single exceptional divisor, and let $\tilde{L} = \pi^* L$ be the pullback of our positive line bundle. This time, we use the commutative diagram

$$\begin{tikzcd} H^0(M, L^k)(p) \arrow[r] \arrow[d, "\cong"] & T_p^{1,0}M \otimes L_p^k \arrow[d, "\cong"] \\ H^0\big(M, \tilde{L}^k \otimes [-E]\big) \arrow[r] & H^0\big(E, i^*\tilde{L}^k \otimes [-E]\big). \end{tikzcd}$$

Note that the restriction of $\tilde{L}^k \otimes [-E]$ to the exceptional divisor is isomorphic to $\mathscr{O}_E(1) \otimes \tilde{L}_p^k$, and so its space of global sections is $H^0\big(E, \mathscr{O}_E(1)\big) \otimes \tilde{L}_p^k$. Sections of $\mathscr{O}_E(1)$ are linear forms in the variables $z_1, \dots, z_n$, which exactly correspond to the holomorphic cotangent space $T_p^{1,0}M$.

In other words, it is now sufficient to prove the surjectivity of

$$H^0\big(\tilde{M}, \tilde{L}^k \otimes [-E]\big) \to H^0\big(E, \mathscr{O}_E(1)\big) \otimes \tilde{L}_p^k,$$

for which we may use the exact sequence

$$H^0\big(\tilde{M}, \tilde{L}^k \otimes [-E]\big) \to H^0\big(E, i^*\tilde{L}^k \otimes [-E]\big) \to H^1\big(\tilde{M}, \tilde{L}^k \otimes [-E]^2\big).$$

To prove the vanishing of the group $H^1\big(\tilde{M}, \tilde{L}^k \otimes [-E]^2\big)$, we argue as before to obtain

$$\tilde{L}^k \otimes [-E]^2 \simeq K_{\tilde{M}} \otimes Q_k$$

for a positive line bundle Q_k, once $k \geq (n-1)m_0 + \ell$. The required vanishing then follows from Theorem 22.2. Again, note that the lower bound on k may depend on the point $p \in M$.

Step 4. To finish the proof, we have to argue that there is a single integer k_0, such that (a) and (b) hold for all points $p, q \in M$ once $k \geq k_0$. We shall prove this by using the compactness of the product $M \times M$.

Recall that (b) holds at some point $p_0 \in M$ iff the differential of the mapping φ_{L^k} is injective. By basic calculus, this implies that φ_{L^k} is injective in a small neighborhood of p_0, and so (a) and (b) are both true for all (p, q) with $p \neq q$ that belong to a small neighborhood of $(p_0, p_0) \in M \times M$. On the other hand, Step 3 shows that (a) holds in a neighborhood of every pair (p, q) with $p \neq q$. It follows that we can cover $M \times M$ by open subsets V_i, on each of which (a) and (b) are true once $k \geq k_i$. By compactness, finitely many of those open sets cover the product, and so we again obtain a single value of k_0 such that φ_{L^k} is an embedding for $k \geq k_0$. This completes the proof of the Kodaira embedding theorem.

Consequences of Kodaira's theorem. In algebraic geometry, a line bundle is called *very ample* if φ_L is an embedding; L is called *ample* if L^k is very ample for $k \gg 0$. Thus what we have shown is: a line bundle L on a compact Kähler manifold M is positive iff it is ample. Thus for the complex geometer, ampleness corresponds to positivity of curvature, in the sense that $\frac{i}{2\pi}\Theta$ is a positive form.

Example 24.1. During the proof of Theorem 23.3, we saw that if $\pi \colon \mathrm{Bl}_p M \to M$ is the blow-up of M at some point p, and if L is a positive line bundle on M, then $\pi^* L^k \otimes [-E]$ is a positive line bundle on $\mathrm{Bl}_p M$ for $k \gg 0$. It follows that if the manifold M is projective, the blow-up $\mathrm{Bl}_p M$ is also projective. Since the latter was defined by gluing, this is not at all obvious.

The Kodaira embedding theorem can be restated to provide a purely cohomological criterion for a compact Kähler manifold to be projective.

Proposition 24.2. *Let M be a compact Kähler manifold. Then M is projective if, and only if, there exists a closed positive $(1,1)$-form $\omega \in A^2(M)$ whose cohomology class $[\omega]$ is rational, i.e., belongs to the subspace $H^2(M,\mathbb{Q}) \subseteq H^2(M,\mathbb{C})$.*

Proof. If M is projective, then we can take for ω the restriction of the Fubini-Study form from projective space. We will prove the converse by showing that M has a positive line bundle. After multiplying ω by a positive integer, we can assume that $[\omega]$ belongs to the image of the map $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{C})$. As M is Kähler, we have $H^2(M,\mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$, and as previously explained, the exact sequence

$$H^1(M,\mathscr{O}_M) \to H^1(M,\mathscr{O}_M^*) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \to H^2(M,\mathscr{O}_M)$$

shows that $[\omega]$ is the first Chern class of a holomorphic line bundle L on M. By construction, L is positive (since its first Chern class is represented by the positive form ω), and so M is projective by Theorem 23.3. $\square$

In certain cases, the criterion can be used directly to prove projectivity. A very useful one is the following.

Corollary 24.3. *If a compact Kähler manifold M satisfies $H^2(M,\mathscr{O}_M) \simeq 0$, then it is necessarily projective.*

Proof. Fix some Kähler metric h_0 on M, and let ω_0 be the Kähler form. Then ω_0 is a closed positive $(1,1)$-form whose cohomology class belongs to $H^2(M,\mathbb{R})$. We can represent classes in $H^2(M,\mathbb{C})$ uniquely by harmonic forms (with respect to the metric h_0), with classes in $H^2(M,\mathbb{R})$ represented by real forms. Moreover, the inner product $(\alpha,\beta)_M$ that we previously defined gives us a way to measure distances in $H^2(M,\mathbb{C})$. By assumption, the two subspaces $H^{0,2}(M)$ and $H^{2,0}(M)$ in the Hodge decomposition are both zero, and so $H^2(M,\mathbb{C}) = H^{1,1}(M)$. In particular, any real harmonic form in $\mathcal{H}^2(M)$ has type $(1,1)$. The space of rational classes $H^2(M,\mathbb{Q})$ is dense in $H^2(M,\mathbb{R})$, and so for any $\varepsilon > 0$, there exists a harmonic $(1,1)$-form ω with rational cohomology class satisfying $\|\omega - \omega_0\|_M < \varepsilon$. Now the point is that, M being compact, any such ω that is sufficiently close to ω_0 will still be positive. Here is a careful proof.[2] Let us choose an orthonormal basis $\omega_0, \omega_1, \dots, \omega_m \in \mathcal{H}^2(M)$, where ω_0 is our initial Kähler form and $\omega_1, \dots, \omega_m$ are real and primitive. We can then write any real harmonic form $\omega \in \mathcal{H}^2(M)$ uniquely as

$$\omega = c_0\omega_0 + c_1\omega_1 + \cdots + c_m\omega_m,$$

with $c_0, \dots, c_m \in \mathbb{R}$. Then

$$\|\omega - \omega_0\|_M^2 = (c_0 - 1)^2 + c_1^2 + \cdots + c_m^2 < \varepsilon$$

implies that $\omega = \omega_0 + (c_0 - 1)\omega_0 + c_1\omega_1 + \cdots + c_m\omega_m$ is the sum of a positive form and a form whose pointwise values are very small, and is therefore positive for sufficiently small values of $\varepsilon > 0$. We can then conclude by the criterion in Proposition 24.2. $\square$

Example 24.4. A *Calabi-Yau manifold* is a compact Kähler manifold M whose canonical bundle K_M is isomorphic to the trivial line bundle, and on which the cohomology groups $H^q(M,\mathscr{O}_M)$ for $1 \leq q \leq \dim M - 1$ vanish. If $\dim M \geq 3$, then such an M can always be embedded into projective space.

[2] I thank Jiasheng for asking about this point.

Example 24.5. Any compact Riemann surface is projective. (This can of course be proved more easily by other methods.)

Complex tori. A nice class of compact Kähler manifolds is that of complex tori, which meant quotients of the form $T = \mathbb{C}^n/\Lambda$, for Λ a lattice in $\mathbb{C}^n$. In the exercises, we have seen that the standard metric on V descends to a Kähler metric on T. To illustrate the usefulness of Kodaira's theorem, we shall settle the following question: when is a complex torus T projective?

Example 24.6. Everyone knows that elliptic curves (the case $n = 1$) can always be embedded into $\mathbb{P}^2$ as cubic curves.

The following theorem, known as Riemann's criterion, gives a necessary and sufficient condition for T to be projective. Since the proof I gave in class was not completely satisfactory, I have added some details here.

Theorem 24.7. *Let $T = \mathbb{C}^n/\Lambda$ be a complex torus. Then T is projective if, and only if, there exists a positive definite Hermitian bilinear form $h\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, whose imaginary part $E = -\operatorname{Im} h$ takes integral values on $\Lambda \times \Lambda$.*

Proof. In order to prove that T is projective, it is enough to find a closed positive $(1,1)$-form whose cohomology class is integral. The two conditions in the proposition are saying that T carries such a form, although some translation is needed to see that this is the case.

Let us first show why the existence of the form h implies that T is projective. Choose a basis $v_1, \dots, v_g \in V$, and let $z_1, \dots, z_g$ be the corresponding linear coordinate system on V. Then $dz_1, \dots, dz_n \in A^{1,0}(T)$ and $d\bar{z}_1, \dots, d\bar{z}_n \in A^{0,1}(T)$ are well-defined smooth forms on T. The positive definite Hermitian bilinear form h is represented by a $g \times g$-matrix with entries

$$h_{j,k} = h(v_j, v_k);$$

it is Hermitian symmetric and positive definite. The associated $(1,1)$-form

$$\omega = \frac{i}{2} \sum_{j,k=1}^{g} h_{j,k} dz_j \wedge d\bar{z}_k$$

is therefore positive and, obviously, closed. To see that its class in $H^2(T, \mathbb{C})$ belongs to the image of $H^2(T, \mathbb{Z})$, we can compute its integrals over a collection of 2-cycles that generate $H_2(T, \mathbb{Z})$. We can use the images in T of

$$[0,1] \times [0,1] \to V, \quad (x, y) \mapsto x\gamma + y\delta,$$

where $\gamma, \delta \in \Gamma$ are two arbitrary elements. Then

$$dz_j \wedge d\bar{z}_k = (\gamma_j \, dx + \delta_j \, dy) \wedge (\overline{\gamma_k} \, dx + \overline{\delta_k} \, dy) = \big(\gamma_j \overline{\delta_k} - \delta_j \overline{\gamma_j}\big) \, dx \wedge dy$$

The integral in question thus becomes

$$\frac{i}{2} \sum_{j,k=1}^{g} \int_0^1 \int_0^1 h_{j,k} dz_j \wedge d\bar{z}_k = \frac{i}{2} \sum_{j,k=1}^{g} h_{j,k} \big(\gamma_j \overline{\delta_k} - \delta_j \overline{\gamma_j}\big) = \frac{i}{2} \Big(h(\gamma, \delta) - h(\delta, \gamma)\Big).$$

This is easily seen to equal $E(\gamma, \delta) = -\operatorname{Im} h(\gamma, \delta)$, and so we get the result.

We also need to prove the converse. Suppose that $\omega \in A^{1,1}(T)$ is a closed positive real $(1,1)$-form whose class in $H^2(T,\mathbb{R})$ lies in the subspace $H^2(T,\mathbb{Z})$. Using $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$, we can write ω as a finite sum

$$\omega = \frac{i}{2} \sum_{j,k=1}^{g} f_{j,k} dz_j \wedge d\bar{z}_k,$$

where each $f_{j,k} \in A(T,\mathbb{C})$ is now a smooth function on T. Positivity of ω means that the matrix with entries $f_{j,k}$ is positive definite at every point of T. We want a matrix with constant entries, and so we consider the averages

$$h_{j,k} = \frac{1}{\operatorname{vol}(T)} \int_T f_{j,k} \operatorname{vol}(g) \in \mathbb{C},$$

where $\operatorname{vol}(g)$ is the volume form for the Euclidean metric on T. The matrix with entries $h_{j,k}$ is then the average of the matrices with entries $f_{j,k}$, and as such, it is still positive definite. It remains to show that the cohomology class of the new form

$$\omega_0 = \frac{i}{2} \sum_{j,k=1}^{g} h_{j,k} dz_j \wedge d\bar{z}_k$$

is still integral. This can be done as follows. For any $x \in T$, consider the translation automorphism $t_x \colon T \to T$, $t_x(y) = x + y$. The cohomology class of the pullback $t_x^*\omega$ belongs to $H^2(T,\mathbb{Z})$, and so the integral

$$\int_{\gamma,\delta} t_x^*\omega$$

over the above generators of $H_2(T,\mathbb{Z})$ takes values in $\mathbb{Z}$ for every $\gamma, \delta \in \Gamma$. It is easy to see that

$$\int_{\gamma,\delta} \omega_0 = \frac{1}{\operatorname{vol}(T)} \int_T \left(\int_{\gamma,\delta} t_x^*\omega \right) \operatorname{vol}(g),$$

and as an average of integers, this is itself an integer. $\square$

Note. If we denote by $J \colon V \to V$ the homomorphism given by multiplication by i, then the fact that h is a Hermitian form implies $h(Jv, Jw) = h(v,w)$. It follows that $E(Jv, Jw) = E(v,w)$; moreover,

$$h(v,w) = E(v, Jw) - iE(v,w),$$

and so E uniquely determines h.