Class 23. The Kodaira embedding theorem (November 19)

Recall that every complex submanifold of projective space is a Kähler manifold: a Kähler metric is obtained by restricting the Fubini-Study to the submanifold. Our next goal is to describe exactly which compact Kähler manifolds are *projective*, i.e., can be embedded into projective space as submanifolds. A necessary condition for M to be projective is the existence of a positive line bundle; indeed, if $M \subseteq \mathbb{P}^N$ is a submanifold, then the restriction of $\mathscr{O}_{\mathbb{P}^N}(1)$ to M is clearly a positive line bundle, since its first Chern class is represented by the restriction of ω_{FS} to M. That this condition is also sufficient is the content of the famous *Kodaira embedding theorem*: a compact complex manifold is projective if and only if it possesses a positive line bundle. In the next two lectures, we will use the Kodaira vanishing theorem to prove this result.

Maps to projective space. We begin by looking at the relationship between holomorphic line bundles and maps to projective space. Suppose then that we have a holomorphic map $f\colon M \to \mathbb{P}^N$ from a compact complex manifold to projective space. We say that f is *nondegenerate* if the image $f(M)$ is not contained in any hyperplane of $\mathbb{P}^N$. It is clearly sufficient to understand nondegenerate maps, because a degenerate map is really a map from M into a projective space of smaller dimension.

On $\mathbb{P}^N$, we have the line bundle $\mathscr{O}_{\mathbb{P}^N}(1)$, which we defined as the dual of the tautological bundle $\mathscr{O}_{\mathbb{P}^N}(-1)$. With respect to the standard open cover of $\mathbb{P}^N$ by the open sets $U_j = \{ [z] \in \mathbb{P}^N \mid z_j \neq 0 \}$, it is described by the transition functions $g_{j,k} = z_k/z_j$. The space of global sections

$$H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(-1)\big) \cong \mathbb{C}[z_0, \dots, z_N]_1$$

is isomorphic to the space of homogeneous polynomials of degree 1, and therefore has dimension $N+1$. (A linear polynomial $L(z) = a_0 z_0 + \dots + a_N z_N$ defines a global section of $\mathscr{O}_{\mathbb{P}^N}(1)$ that is represented on the open set U_j by the holomorphic function $L(z)/z_j$; alternatively, L defines a linear functional on the vector space $\mathbb{C}^{N+1}$, and therefore on each fiber on the tautological line bundle.)

Given a nondegenerate map $f\colon M \to \mathbb{P}^N$, we obtain a holomorphic line bundle $L = f^*\mathscr{O}_{\mathbb{P}^N}(1)$, the *pullback* of $\mathscr{O}_{\mathbb{P}^N}(1)$ via the map f. In general, assuming that $f\colon X \to Y$ is a holomorphic mapping between two complex manifolds, and $\pi\colon E \to Y$ a holomorphic vector bundle on Y, the pullback bundle

$$f^*E = \{ (y, e) \in Y \times E \mid f(y) = \pi(e) \}$$

sits in the following commutative diagram:

$$\begin{array}{ccc} f^*E & \longrightarrow & X \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\ f\ } & X \end{array}$$

The fiber of f^*E at a point $y \in Y$ is therefore exactly the fiber of the original bundle E at the point $f(y)$. In our specific case, we have $L_p = \mathscr{O}_{\mathbb{P}^N}(1)_{f(p)}$. More concretely, we may define L as being the line bundle with transition functions $g_{j,k} \circ f$ on the cover of M by the $N+1$ open sets $f^{-1}(U_j)$. Now every section of $\mathscr{O}_{\mathbb{P}^N}(1)$ defines, by pulling back, a section of L on M, and the resulting map

$$H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1)\big) \to H^0(M, L)$$

is injective since f is nondegenerate. We have $\dim H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1)\big) = N+1$.

Conversely, suppose that we have a holomorphic line bundle L on M, together with a subspace $V \subseteq H^0(M,L)$ that is *base-point free*. By this we mean that at every point $p \in M$, there should be a holomorphic section $s \in V$ that does not vanish at the point p (and hence generates the one-dimensional vector space L_p). We can then construct a holomorphic mapping from M to projective space as follows: Let $N = \dim V - 1$, choose a basis $s_0, s_1, \dots, s_N \in V$, and define

$$f\colon M \to \mathbb{P}^N, \quad f(p) = \big[s_0(p), s_1(p), \dots, s_N(p)\big].$$

That is to say, at each point of M, at least one of the sections, say s_0, is nonzero; in some neighborhood U of the point, we can then $s_j = f_j s_0$ for $f_j \in \mathscr{O}_M(U)$ holomorphic. On that open set U, the mapping f is then given by the formula $f(p) = [1, f_1(p), \dots, f_N(p)] \in \mathbb{P}^N$.

Note. A more invariant description of the map f is the following: Let $\mathbb{P}(V)$ be the set of codimension 1 subspaces of V; any such is the kernel of a linear functional on V, unique up to scaling, and so $\mathbb{P}(V)$ is naturally isomorphic to the projective space of lines through the origin in V^*. From this point of view, the mapping $f\colon M \to \mathbb{P}(V)$ takes a point $p \in M$ to the subspace $V(p) = \big\{\, s \in V \;\big|\; s(p) = 0 \,\big\}$. Since V is assumed to be base-point free, $V(p) \subseteq V$ is always of codimension 1, and so the mapping is well-defined.

The two processes above are clearly inverse to each other, and so we obtain the following result: nondegenerate holomorphic mappings $f\colon M \to \mathbb{P}^N$ are in one-to-one correspondence with base-point free subspaces $V \subseteq H^0(M,L)$ of dimension $N+1$. In particular, any holomorphic line bundle L whose space of global sections $H^0(M,L)$ is base-point free defines a holomorphic mapping

$$\varphi_L\colon M \to \mathbb{P}^N,$$

where $N = \dim H^0(M,L) - 1$. We abbreviate this by saying that L is base-point free; alternatively, one says that L is *globally generated*, since it implies that the restriction mapping $H^0(M,L) \to L_p$ is surjective for each point $p \in M$.

Example 23.1. Consider the line bundle $\mathscr{O}_{\mathbb{P}^1}(k)$ on the Riemann sphere $\mathbb{P}^1$. We have seen in the exercises that its space of sections is isomorphic to the space of homogeneous polynomials of degree k in $\mathbb{C}[z_0, z_1]$. What is the corresponding map to projective space? If we use the monomials $z_0^k, z_0^{k-1}z_1, \dots, z_0 z_1^{k-1}, z_1^k$ as a basis, we see that the line bundle is base-point free, and that the map is given by

$$\mathbb{P}^1 \to \mathbb{P}^k, \quad [z_0, z_1] \mapsto [z_0^k, z_0^{k-1}z_1, \dots, z_0 z_1^{k-1}, z_1^k].$$

It is easy to see that this is an embedding; the image is the so-called *rational normal curve* of degree k.

Example 23.2. More generally, the line bundle $\mathscr{O}_{\mathbb{P}^n}(k)$ embeds $\mathbb{P}^n$ into the larger projective space $\mathbb{P}^N$, where $N = \binom{n+k}{n} - 1$; this is the so-called *Veronese embedding*.

The Kodaira embedding theorem. For a line bundle L and a positive integer k, we let $L^k = L \otimes L \otimes \cdots \otimes L$ be the k-fold tensor product of L. We can now state Kodaira's theorem in a more precise form.

Theorem 23.3. *Let M be a compact complex manifold, and let L be a positive line bundle on M. Then there is a positive integer k_0 with the following property: for* *every $k \geq k_0$, the line bundle L^k is base-point free, and the holomorphic mapping φ_{L^k} is an embedding of M into projective space.*

In general, suppose that L is a base-point free line bundle on M; let us investigate under what conditions the corresponding mapping $\varphi\colon M \to \mathbb{P}^N$ is an embedding. Clearly, the following two conditions are necessary and sufficient:

(a) φ is injective: if $p, q \in M$ are distinct points, then $\varphi(p) \neq \varphi(q)$.
(b) At each point $p \in M$, the differential $\varphi_*\colon T'_p M \to T'_{\varphi(p)}\mathbb{P}^N$ is injective.

Indeed, since M is compact, the map φ is automatically open, and so the first condition implies that φ is a homeomorphism onto its image $\varphi(M)$. The second condition, together with the implicit function theorem, can then be used to show that the inverse map φ^{-1} is itself holomorphic, and hence that φ is an embedding.

We shall now put both conditions in a more intrinsic form that only refers to the line bundle L and its sections. As above, let $s_0, s_1, \dots, s_N$ be a basis for the space of sections $H^0(M, L)$. Then (a) means that, for any two distinct points $p, q \in M$, the two vectors $\big(s_0(p), s_1(p), \dots, s_N(p)\big)$ and $\big(s_0(q), s_1(q), \dots, s_N(q)\big)$ should be linearly independent. Equivalently, the restriction map

$$H^0(M, L) \to L_p \oplus L_q$$

that associates to a section s the pair of values $(s(p), s(q))$ should be surjective. If this is satisfied, one says that L *separates points*.

Consider now the other condition. Fix a point $p \in M$, and suppose for simplicity that $s_0(p) \neq 0$ and $s_1(p) = \cdots = s_N(p) = 0$. In a neighborhood of p, we then have $s_j = f_j s_0$ for holomorphic functions $f_1, \dots, f_N$ that vanish at the point p, and (b) is saying that the matrix of partial derivatives

$$\begin{pmatrix} \partial f_1/\partial z_1 & \partial f_1/\partial z_2 & \cdots & \partial f_1/\partial z_n \\ \partial f_2/\partial z_1 & \partial f_2/\partial z_2 & \cdots & \partial f_2/\partial z_n \\ \vdots & \vdots & & \vdots \\ \partial f_N/\partial z_1 & \partial f_N/\partial z_2 & \cdots & \partial f_N/\partial z_n \end{pmatrix}$$

should have rank n at the point p. Another way to put this is that the holomorphic 1-forms $df_1, df_2, \dots, df_N$ should span the holomorphic cotangent space $T^{1,0}_p M$. More intrinsically, we let $H^0(M, L)(p)$ denote the space of sections that vanish at p. We can write any such section as $s = f s_0$, with f holomorphic in a neighborhood of p and satisfying $f(p) = 0$. Then $df(p) \otimes s_0$ is a well-defined element of the vector space $T^{1,0}_p M \otimes L_p$, independent of the choice of s_0; in these terms, condition (b) is equivalent to the surjectivity of the linear map

$$H^0(M, L)(p) \to T^{1,0}_p M \otimes L_p.$$

If this holds, one says that L *separates tangent vectors*.

Since our main tool is a vanishing theorem, it is useful to notice that both conditions can also be stated using the language of sheaves. For any point $p \in M$, we define $\mathscr{I}_p$ as the sheaf of all holomorphic functions on M that vanish at the point p. Likewise, we let $\mathscr{I}_p(L)$ denote the sheaf of holomorphic sections of L that vanish at p, and note that it is a subsheaf of the sheaf $\mathscr{O}_M(L)$ of all holomorphic sections of L. We then have an exact sequence of sheaves

$$0 \to \mathscr{I}_p(L) \to \mathscr{O}_M(L) \to L_p \to 0,$$

where we consider L_p as a sheaf supported at the point p (meaning that for any open set $U \subseteq M$, we have $L_p(U) = L_p$ if $p \in U$, and zero otherwise). The relevant portion of the long exact sequence of cohomology groups is

$$0 \to H^0\big(M, \mathscr{I}_p(L)\big) \to H^0(M, L) \to L_p \to H^1\big(M, \mathscr{I}_p(L)\big),$$

and so the surjectivity of the restriction map would follow from the vanishing of the group $H^1\big(M, \mathscr{I}_p(L)\big)$. The problem is that, unless M is a Riemann surface, this is not the cohomology group of a holomorphic line bundle, and so the Kodaira vanishing theorem does not apply to it. To overcome this difficulty, we shall use the device of blowing up: it replaces a point (codimension n) with a copy of $\mathbb{P}^{n-1}$ (codimension $n-1$), and thus allows us to work with line bundles.

Blowing up. Let M be a complex manifold of dimension n. Recall from Lecture 5 that the blow-up of M at a point p is another complex manifold $\operatorname{Bl}_p M$, in which the point is replaced by a copy of $\mathbb{P}^{n-1}$. This so-called exceptional divisor E is basically the projective space of lines in $T'_p M$, and should be thought of as parametrizing directions from p into M. Here is a brief review of the construction of $\operatorname{Bl}_p M$. First, we defined the blow-up of $\mathbb{C}^n$ at the origin as

$$\operatorname{Bl}_0 \mathbb{C}^n = \big\{ \big(z, [a]\big) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \bigm| z \text{ lies on the line } \mathbb{C} \cdot a \big\}.$$

The first projection $\pi\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$ is an isomorphism outside the origin, and $\pi^{-1}(0)$ is a copy of $\mathbb{P}^{n-1}$. For any open set $D \subseteq \mathbb{C}^n$ containing the origin, we then define $\operatorname{Bl}_0 D$ as $\pi^{-1}(D)$. Finally, given a point p on an arbitrary complex manifold M, we choose a coordinate chart $\phi\colon U \to D$ around p, with $D \subseteq \mathbb{C}^n$ an open polydisk, and construct the complex manifold $\operatorname{Bl}_p M$ by gluing together $M \setminus \{p\}$ and $\operatorname{Bl}_0 D$ according to the map ϕ.

We now have to undertake a more careful study of the blow-up. From now on, we set $\tilde{M} = \operatorname{Bl}_p M$, and let $\pi\colon \operatorname{Bl}_p M \to M$ be the blow-up map. The exceptional divisor $E = \pi^{-1}(p)$ is a complex submanifold of $\tilde{M}$ of dimension $n-1$. We briefly recall why. The statement only depends on a small open neighborhood of E in $\tilde{M}$, and so it suffices to prove this for the exceptional divisor in $\operatorname{Bl}_0 \mathbb{C}^n$. Here, we have the second projection $q\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{P}^{n-1}$, and so we get n natural coordinate charts $V_j = q^{-1}(U_j)$ (where U_j is the set of points $[a] \in \mathbb{P}^{n-1}$ with $a_j \neq 0$). These are given by

$$\mathbb{C}^n \to V_j, \quad (b_1, \dots, b_n) \mapsto \big(b_j a, [a]\big)$$

where $a = (b_1, \dots, b_{j-1}, 1, b_{j+1}, \dots, b_n)$. In these charts, the map π takes the form

$$\pi(b_1, \dots, b_n) = \big(b_j b_1, \dots, b_j b_{j-1}, b_j, b_j b_{j+1}, \dots, b_j b_n\big),$$

and so the exceptional divisor $E \cap U_j$ is exactly the submanifold defined by the single equation $b_j = 0$.

Since E has dimension $n-1$, it determines a holomorphic line bundle $\mathscr{O}_{\tilde{M}}(-E)$, whose sections over any open set $U \subseteq \tilde{M}$ are those holomorphic functions on U that vanish along $U \cap E$. To simplify the notation, we write $\mathscr{O}_E(1)$ for the image of $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ under the isomorphism $E \simeq \mathbb{P}^{n-1}$.

Lemma 23.4. *The restriction of $\mathscr{O}_{\tilde{M}}(-E)$ to the exceptional divisor is isomorphic to $\mathscr{O}_E(1)$.*

Proof. The statement only depends on a small neighborhood of E in $\tilde{M}$, and we may therefore assume that we are dealing with the blowup of $\mathbb{C}^n$ at the origin.

We have seen in the exercises that the second projection $q\colon \mathrm{Bl}_0\,\mathbb{C}^n \to \mathbb{P}^{-1}$ is the holomorphic line bundle $\mathscr{O}_{\mathbb{P}^{n-1}}(-1)$. The exceptional divisor is precisely the image of the zero section, and by another exercise, its line bundle is isomorphic to $q^*\mathscr{O}_{\mathbb{P}^{n-1}}(1)$. Obviously, the restriction of this line bundle to the exceptional divisor is now $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$, as claimed. □

To simplify the notation a little, we shall write $[-E]$ for the line bundle $\mathscr{O}_{\tilde{M}}(-E)$, and $[E]$ for its dual. As usual, we also let $[E]^k$ be the k-fold tensor product of $[E]$ with itself. Lastly, we write K_M for the canonical bundle Ω^n_M. In order to apply the Kodaira vanishing theorem on $\tilde{M}$, we need to now how the canonical bundle $K_{\tilde{M}}$ is related to K_M.

Lemma 23.5. *The canonical bundle of $\tilde{M}$ satisfies $K_{\tilde{M}} \simeq \pi^* K_M \otimes [E]^{n-1}$.*

Proof. To show the gist of the statement, we shall only prove this in the case $M = \mathbb{C}^n$ and $\tilde{M} = \mathrm{Bl}_0\,\mathbb{C}^n$. With $z_1, \dots, z_n$ the usual coordinate system on $\mathbb{C}^n$, the canonical bundle Ω^n_M is trivial, generated by the section $dz_1 \wedge \cdots \wedge dz_n$. To prove the lemma, it is enough to show that the line bundle $K_{\tilde{M}} \otimes [-E]^{n-1}$ is trivial on $\tilde{M}$. Note that its holomorphic sections are holomorphic n-forms that vanish at least to order $n-1$ along E.

Consider the pullback $\pi^*(dz_1 \wedge \cdots \wedge dz_n)$. In one of the n open sets V_j that cover the blow-up, the exceptional divisor is defined by the equation $b_j = 0$, and the map π is given by the formula $\pi(b_1, \dots, b_n) = (b_j b_1, \dots, b_j b_{j-1}, b_j, b_j b_{j+1}, \dots, b_j b_n)$. Consequently, we have

$$\begin{aligned}\pi^*(dz_1 \wedge \cdots \wedge dz_n) &= d(b_j b_1) \wedge \cdots \wedge d(b_j b_{j-1}) \wedge db_j \wedge d(b_j b_{j+1}) \wedge \cdots \wedge d(b_j b_n) \\ &= b_j^{n-1} db_1 \wedge \cdots \wedge db_n,\end{aligned}$$

and so $\pi^*(dz_1 \wedge \cdots \wedge dz_n)$ is a global section of $K_{\tilde{M}} \otimes [-E]^{n-1}$. The above formula shows that, moreover, it generates that line bundle on each open set V_j, and so the line bundle is indeed trivial. □

Let L be a positive line bundle on M, and write $\tilde{L} = \pi^* L$ for the pullback line bundle on $\tilde{M}$. Then $\tilde{L}$ is no longer positive: the first Chern class $c_1(\tilde{L})$ is the pulllback of $c_1(L)$, and so it is trivial along the exceptional divisor E; in particular, at cannot be represented by a Kähler form on $\tilde{M}$. The following lemma shows how to fix this problem.

Lemma 23.6. *Let L be a positive line bundle on M. Then for sufficiently large k, the line bundle $\tilde{L}^k \otimes [-E]$ is again positive.*