Class 20. Holomorphic vector bundles and connections (November 7)

Hypersurfaces in projective space, continued.

Example 20.1. Let us look at the case of a *K3-surface*, namely a complex submanifold $M \subseteq \mathbb{P}^3$ defined by a homegeneous equation $F \in \mathbb{C}[z_0, z_1, z_2, z_3]$ of degree $d = 4$. Here we have $H^1(M, \mathbb{C}) \simeq H^1(\mathbb{P}^3, \mathbb{C}) = 0$, and $H^2(M, \mathbb{C}) = H^2(\mathbb{P}^3, \mathbb{C}) \oplus H_0^2(M, \mathbb{C})$. To compute the Hodge decomposition on the primitive part of the cohomology, we apply Griffiths' formula (with $n = 2$ and $d = 4$). Firstly,

$$H_0^{2,0}(M) \simeq R(F)_0 \simeq S_0,$$

and therefore $h^{2,0}(M) = 1$. Secondly,

$$H_0^{1,1}(M) \simeq R(F)_4 = S_4 \Big/ \sum_{j=1}^{3} S_1 \frac{\partial F}{\partial z_j},$$

and counting dimension, we find that

$$h^{1,1}(M) = 1 + \dim S_4 - 4 \dim S_1 = 1 + \binom{4+3}{3} - 4 \cdot \binom{1+3}{3} = 20.$$

Because we know from general principles that $h^{0,2}(M) = h^{2,0}(M)$, those two numbers suffice to write down the Hodge diamond of M, which looks like

$$\begin{matrix} & & \mathbb{C} & & \\ & 0 & & 0 & \\ \mathbb{C} & & \mathbb{C}^{20} & & \mathbb{C} \\ & 0 & & 0 & \\ & & \mathbb{C} & & \end{matrix}$$

Note. If the polynomial F is complicated, counting the dimension of the space $R(F)_{(n+1-p)d-(n+2)}$ may be fairly involved. Luckily, there is a shortcut: One can prove that the dimensions are the same for any irreducible homogeneous polynomial F of degree d (whose zero set is a submanifold), and so it suffices to do the computations in the easy case $F = z_0^d + z_1^d + \cdots + z_{n+1}^d$. The reason is that the space of all such polynomials (as an open subset of a complex vector space) is connected, and that the Hodge numbers $\dim H_0^{p,n-p}$ are continuous functions on that space.

Residues. The proof of Theorem 19.6 requires several results from algebraic geometry and algebraic topology that we do not have at our disposal; but we can at least describe the so-called *residue map*

$$A^{n+1}(M, n+1-p) \to H_0^{p,n-p}(M)$$

that induces the isomorphism. Recall the notion of a residue from complex analysis: given a meromorphic function $f(z)$ on an open set U, holomorphic on $U \setminus \{z_0\}$, we write $f(z) = \sum_{j \in \mathbb{Z}} a_j (z - z_0)^j$ as a Laurent series, and then

$$\operatorname{Res}_{z_0} f = \frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} f(z) dz = a_{-1}.$$

Put differently, the residue map assigns to a meromorphic one-form $f(z)dz$ a complex number at each point where the form has a pole.

The same construction works for $M \subseteq \mathbb{P}^{n+1}$, and explains the case $p = n$ from above: there is a map Res_M from rational $(n+1)$-forms on $\mathbb{P}^{n+1}$ with a first-order pole along M to the space of holomorphic n-forms on M. Namely, around each point of the submanifold M, we can find local coordinates $t_1, \dots, t_{n+1}$ on an open neighborhood $U \subseteq \mathbb{P}^{n+1}$ such that $M \cap U$ is the subset defined by $t_1 = 0$. Given $\alpha \in A^{n+1}(M, 1)$, we then have

$$\alpha|_U = \frac{f(t_1, t_2, \dots, t_{n+1})}{t_1} dt_1 \wedge \dots \wedge dt_{n+1}$$

for some holomorphic function f, and then the residue of $\alpha|_U$ is the holomorphic n-form $f(0, t_2, \dots, t_{n+1}) dt_2 \wedge \dots \wedge dt_{n+2}$ on $M \cap U$. One can show that this does not depend on the choice of coordinates, and thus defines a global holomorphic n-form $\operatorname{Res}_M \alpha$ on M.

On forms with a pole of higher order, an additional step is needed. Suppose that $\alpha \in A^{n+1}(M, \ell)$ has a pole of order at most $\ell \geq 2$. In local coordinates, we can again express α in the form

$$\alpha|_U = \frac{f(t_1, t_2, \dots, t_{n+1})}{t_1^\ell} dt_1 \wedge \dots \wedge dt_{n+1}.$$

For $\ell \geq 2$, the identity

$$d\left(\frac{f}{t_1^{\ell-1}} dt_2 \wedge \dots \wedge dt_{n+1}\right) = -(\ell - 1)\alpha|_U + \frac{\partial f/\partial t_1}{t_1^{\ell-1}} dt_1 \wedge \dots \wedge dt_{n+1}$$

allows us to write $\alpha|_U = \beta + d\gamma$, where β is an $(n+1)$-form, and γ an n-form, both holomorphic on $U \setminus (M \cap U)$ and with a pole of order at most $\ell - 1$. In other words, we can adjust $\alpha|_U$ by an exact form and reduce the order of the pole. To do this globally, choose an open covering $\mathbf{U}$ of M by suitable open subsets of $\mathbb{P}^{n+1}$, and let $1 = \sum_{i \in I} \rho_i$ be a partition of unity subordinate to $\mathbf{U}$. If $\alpha|_{U_i} = \beta_i + d\gamma_i$, then

$$\alpha = \sum_{i \in I} \rho_i \alpha|_{U_i} = d\left(\sum_{i \in I} \rho_i \gamma_i\right) + \sum_{i \in I} \big(\rho_i \beta_i - d\rho_i \wedge \gamma_i\big).$$

On the principle that the residue of an exact form should be zero, we can thus replace α by the second term on the right-hand side, which is a $(n+1)$-form α_1 with smooth coefficients and a pole of order at most $\ell - 1$. Note that the type of α_1 is now $(n+1, 0) + (n, 1)$, since $d\rho_i$ is no longer a holomorphic form. Continuing in this manner, we can reduce the order of the pole step-by-step until we arrive at a first-order pole where we know how to define the residue.

If we apply the above process to $\alpha \in A^{n+1}(M, n+1-p)$, then α_1 will have a pole of order at most $(n-p)$ and be of type $(n+1, 0) + (n, 1)$; eventually, we arrive at α_{n-p} which has a first-order pole and is of type $(n+1, 0) + \dots + (p+1, n-p)$. If we take the residue (by looking at the coefficient of dt_1/t_1 in local coordinates), we thus obtain

$$\operatorname{Res}_M \alpha \underset{\text{def}}{=} \operatorname{Res}_M \alpha_{n-p} \in A^{n,0}(M) \oplus \dots \oplus A^{p,n-p}(M),$$

and this explains why poles of higher order give rise to forms of different types on M. One can show that the resulting form is closed and independent of the choices made; in this way, we obtain the map

$$\operatorname{Res}_M \colon \frac{A^{n+1}(M, n+1-p)}{A^{n+1}(M, n-p) + dA^n(M, n-p)} \to H_0^{p,n-p}(M).$$

The proof that it is an isomorphism is nontrivial.

Holomorphic vector bundles. Let M be a complex manifold. Recall that a holomorphic vector bundle of rank r is a complex manifold E, together with a holomorphic mapping $\pi\colon E \to M$, such that two conditions are satisfied:

(1) For each point $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is a $\mathbb{C}$-vector space of dimension r.
(2) For every $p \in M$, there is an open neighborhood U and a biholomorphism
$$\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^r$$
mapping E_p into $\{p\} \times \mathbb{C}^r$, such that the composition $E_p \to \{p\} \times \mathbb{C}^r \to \mathbb{C}^r$ is an isomorphism of $\mathbb{C}$-vector spaces.

For two local trivializations (U_α, ϕ_α) and (U_β, ϕ_β), the composition $\phi_\alpha \circ \phi_\beta^{-1}$ is of the form $(\mathrm{id}, g_{\alpha,\beta})$ for a holomorphic mapping
$$g_{\alpha,\beta}\colon U_{\alpha,\beta} \to \mathrm{GL}_r(\mathbb{C}).$$
As we have seen, these transition functions satisfy the compatibility conditions
$$\begin{aligned} g_{\alpha,\beta} \cdot g_{\beta,\gamma} \cdot g_{\gamma,\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma, \\ g_{\alpha,\alpha} &= \mathrm{id} \quad \text{on } U_\alpha; \end{aligned}$$
conversely, every collection of transition functions determines a holomorphic vector bundle. Also recall that a holomorphic section of the vector bundle is a holomorphic mapping $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}$; locally, such a section is described by holomorphic functions $s_\alpha\colon U_\alpha \to \mathbb{C}^r$, subject to the condition that $g_{\alpha,\beta} \cdot s_\beta = s_\alpha$ on $U_\alpha \cap U_\beta$.

Definition 20.2. A *morphism* between two holomorphic vector bundles $\pi\colon E \to M$ and $\pi'\colon E' \to M$ is a holomorphic mapping $f\colon E \to E'$ satisfying $\pi' \circ f = \pi$, such that the restriction of f to each fiber is a linear map $f_p\colon E_p \to E'_p$. If each f_p is an isomorphism of vector spaces, then f is said to be an *isomorphism*.

Example 20.3. The trivial vector bundle of rank r is the product $M \times \mathbb{C}^r$. A vector bundle E is trivial if it is isomorphic to the trivial bundle. Equivalently, E is trivial if it admits r holomorphic sections $s_1, \dots, s_r$ whose values $s_1(p), \dots, s_r(p)$ give a basis for the vector space E_p at each point $p \in M$.

Given a holomorphic vector bundle $\pi\colon E \to M$, we let $A(U, E)$ denote the space of smooth sections of E over an open set $U \subseteq M$. Likewise, $A^{p,q}(U, E)$ denotes the space of (p, q)-forms with coefficients in E; in a local trivialization $\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$, these are given by r-tuples $\omega_\alpha \in A^{p,q}(U)^{\oplus r}$, subject to the relation
$$\omega_\alpha = g_{\alpha,\beta} \cdot \omega_\beta$$
on $U_\alpha \cap U_\beta$. As usual, they can also be viewed as sections of a sheaf $\mathscr{A}^{p,q}(E)$.

Example 20.4. Say L is a line bundle (so $r = 1$), which means that the transition functions $g_{\alpha,\beta} \in \mathscr{O}_M^*(U_\alpha \cap U_\beta)$ are holomorphic functions. In this case, a (p, q)-form with coefficients in L is nothing but a collection of smooth forms $\omega_\alpha \in A^{p,q}(U_\alpha)$, subject to the condition that $\omega_\alpha = g_{\alpha,\beta}\omega_\beta$. The individual forms do not agree on the intersections between the open sets (as they would for a usual (p, q)-form), but differ by the factor $g_{\alpha,\beta}$. One can view this as a “twisted” version of (p, q)-forms.

Hermitian metrics and the Chern connection. For a smooth function $f \in A(U)$, the exterior derivative df is a smooth 1-form on U. Since M is a complex manifold, we have $d = \partial + \bar{\partial}$, and correspondingly, $df = \partial f + \bar{\partial} f$. Because of the Cauchy-Riemann equations, f is holomorphic if and only if $\bar{\partial} f \in A^{0,1}(U)$ is zero.

For a holomorphic vector bundle $E \to M$, there similarly exists an operator $\bar{\partial} \colon A(M,E) \to A^{0,1}(M,E)$, with the property that a smooth section s is holomorphic iff $\bar{\partial} s = 0$. To construct this $\bar{\partial}$-operator, note that in a local trivialization $\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$, smooth sections of E are given by smooth mappings $s_\alpha \colon U_\alpha \to \mathbb{C}^r$; we may then define $\bar{\partial} s_\alpha = (\bar{\partial} s_{\alpha,1}, \dots, \bar{\partial} s_{\alpha,r})$, which is a vector of length r whose entries are $(0,1)$-forms. On the overlap $U_\alpha \cap U_\beta$ between two trivializations, we have $s_\alpha = g_{\alpha,\beta} \cdot s_\beta$, and therefore

$$\bar{\partial} s_\alpha = g_{\alpha,\beta} \cdot \bar{\partial} s_\beta$$

because the entries of the $r \times r$-matrix $g_{\alpha,\beta}$ are holomorphic functions. This shows that if $s \in A(U,E)$, then $\bar{\partial} s$ is a well-defined element of $A^{0,1}(U,E)$.

On the other hand, this method cannot be used to define analogues of d or ∂, because the corresponding derivatives of the $g_{\alpha,\beta}$ do not vanish. The correct generalization of d, as it turns out, is that of a connection on E. As in differential geometry, a *connection* on a complex vector bundle is a mapping

$$\nabla \colon T(M) \times A(M,E) \to A(M,E)$$

that associates to a smooth tangent vector field ξ and a smooth section s another smooth section $\nabla_\xi s$, to be viewed as the derivative of s along ξ. The connection is required to be $A(M)$-linear in its first argument and to satisfy the Leibniz rule

$$\nabla_\xi (fs) = (\xi f) \cdot s + f \nabla_\xi s$$

for any smooth function f. Given a local trivialization $\phi \colon \pi^{-1}(U) \to U \times \mathbb{C}^r$, we have r distinguished holomorphic sections $s_1, \dots, s_r$ of E, corresponding to the coordinate vectors on $\mathbb{C}^r$. We can then represent the action of the connection as

$$\nabla s_j = \sum_{k=1}^{r} \theta_{j,k} \otimes s_k$$

for certain $\theta_{j,k} \in A^1(U)$; this shorthand notation means that

$$\nabla_\xi s_j = \sum_{k=1}^{r} \theta_{j,k}(\xi) s_k.$$

Because of the Leibniz rule, the 1-forms $\theta_{j,k}$ uniquely determine the connection.

As in differential geometry, it is necessary to choose a metric on the vector bundle before one has a canonical connection. We have already encountered the following notion for the holomorphic tangent bundle $T'M$.

Definition 20.5. A *Hermitian metric* on a complex vector bundle $\pi \colon E \to M$ is a collection of Hermitian inner products $h_p \colon E_p \times E_p \to M$ that vary smoothly with $p \in M$, in the sense that $h(s_1, s_2)$ is a smooth function for any two smooth sections $s_1, s_2 \in A(M,E)$.

Given a local trivialization $\phi \colon \pi^{-1}(U) \to U \times \mathbb{C}^r$ of the vector bundle as above, we describe the Hermitian metric h through its coefficient matrix, whose entries

$$h_{j,k} = h(s_j, s_k)$$

are smooth functions on U. We have $h_{k,j} = \overline{h_{j,k}}$, and the matrix is positive definite.

It turns out that, once we have chosen a Hermitian metric on E, there is a unique connection compatible with the metric and the complex structure on E. To define it, we observe that the complexified tangent bundle splits as $T_{\mathbb{C}}M = T'M \oplus T''M$ into the holomorphic and antiholomorphic tangent bundles. Correspondingly, we can split any connection on E as $\nabla = \nabla' + \nabla''$, with $\nabla' \colon T'(M) \times A(M,E) \to A(M,E)$ and $\nabla'' \colon T''(M) \times A(M,E) \to A(M,E)$.

Proposition 20.6. *Let E be a holomorphic vector bundle with a Hermitian metric h. Then there exists a unique connection that is compatible with the metric, in the sense that for every smooth tangent vector field ξ, we have*

$$\xi \cdot h(s_1, s_2) = h(\nabla_\xi s_1, s_2) + h(s_1, \nabla_\xi s_2),$$

and compatible with the complex structure, in the sense that

$$\nabla''_\xi s = (\bar{\partial} s)(\xi)$$

for any smooth section ξ of the anti-holomorphic tangent bundle $T''M$.

This connection is called the *Chern connection* of the holomorphic vector bundle E; one usually abbreviates the second condition by writing $\nabla'' = \bar{\partial}$.

Proof. To prove the uniqueness, suppose that we have such a connection ∇; we will find a formula for the coefficients $\theta_{j,k}$ in terms of the metric. So let $\phi \colon \pi^{-1}(U) \to U \times \mathbb{C}^r$ be a local trivialization of the vector bundle, and let $s_1, \dots, s_r$ denote the corresponding holomorphic sections of E over U. The Hermitian metric is described by its coefficient matrix, whose entries $h_{j,k} = h(s_j, s_k)$ are smooth functions on U. The second condition means that $\nabla'' s_j = \bar{\partial} s_j = 0$ because each s_j is holomorphic, and so we necessarily have

$$\nabla s_j = \nabla' s_j = \sum_{k=1}^r \theta_{j,k} \otimes s_k$$

with $(1,0)$-forms $\theta_{j,k} \in A^{1,0}(U)$ that uniquely determine the connection. By the first condition,

$$dh_{j,k} = h(\nabla s_j, s_k) + h(s_j, \nabla s_k) = \sum_{l=1}^r \bigl(h_{l,k}\theta_{j,l} + h_{j,l}\overline{\theta_{k,l}}\bigr),$$

and this identity shows that $\partial h_{j,k} = \sum h_{l,k}\theta_{j,l}$ and $\bar{\partial} h_{j,k} = \sum h_{j,l}\overline{\theta_{k,l}}$ (which is the conjugate of the other identity). If we let $h^{j,k}$ denote the entries of the inverse matrix, it follows that

$$\theta_{j,k} = \sum_{l=1}^r h^{l,k} \partial h_{j,l},$$

which proves the uniqueness of the Chern connection. Conversely, we can use this formula to define the connection locally; because of uniqueness, the local definitions have to agree on the intersections of different open sets, and so we get a globally defined connection on E. □

Example 20.7. One should think of the Chern connection ∇ as a replacement for the exterior derivative d, and of ∇' as a replacement for ∂; in this way, the identity $\nabla = \nabla' + \bar{\partial}$ generalizes the formula $d = \partial + \bar{\partial}$. In fact, d is the Chern connection on the trivial bundle $E = M \times \mathbb{C}$ (whose smooth sections are the smooth functions) for the Hermitian metric induced by the standard metric on $\mathbb{C}$.