Class 19. Examples of Kähler manifolds (November 5)

The Hodge index theorem. Last time, we showed that the pairing

$$Q(\alpha, \beta) = (-1)^{k(k-1)/2} \int_M \omega^{n-k} \wedge \alpha \wedge \beta$$

on $H^k(M, \mathbb{C})$ has the property that $i^{p-q} Q(\alpha, \bar{\alpha}) > 0$ for any nonzero primitive cohomology class $\alpha \in H^{p,q}$. The following special case of this fact is very useful in the study of compact Kähler surfaces.

Example 19.1. Let us consider the case of a compact Kähler surface M, where $n = \dim M = 2$. Here the Hodge decomposition takes the form

$$H^2(M, \mathbb{C}) = H^{2,0} \oplus \big(H_0^{1,1} \oplus \mathbb{C}\omega\big) \oplus H^{0,2},$$

with $H_0^{1,1} = \ker(\Lambda \colon H^{1,1} \to H^{0,0})$ the primitive cohomology. According to the bilinear relations, the form $\int_M \alpha \wedge \overline{\beta}$ is positive definite on $\mathbb{C}\omega$ and on the subspace $H^{2,0} \oplus H^{0,2}$; on the other hand, it is negative definite on the primitive subspace $H_0^{1,1}$. Put differently, the quadratic form $Q(\alpha) = \int_M \alpha \wedge \alpha$ has signature $(1, m)$ on the space $H^{1,1}(M) \cap H^2(M, \mathbb{R})$, where $m = \dim H_0^{1,1}$, a result known as the *Hodge index theorem* for surfaces.

We shall now look at several examples of Kähler and non-Kähler manifolds, and compute the Hodge decomposition in a few important examples.

The Hopf surface. The Hodge decomposition shows that compact Kähler manifolds are special (in their topological or cohomological properties), when compared to arbitrary compact complex manifolds. In this section, we construct an example of a compact complex manifold, the so-called *Hopf surface*, that admits no Kähler metric. Let $\mathbb{S}^3$ be the three-sphere in $\mathbb{C}^2$, defined as the set of points (z_1, z_2) such that $|z_1|^2 + |z_2|^2 = 1$. There is a diffeomorphism

$$\varphi \colon \mathbb{S}^3 \times \mathbb{R} \to \mathbb{C}^2 \setminus \{0\}, \quad \varphi(z_1, z_2, t) = \big(e^t z_1, e^t z_2\big).$$

The infinite cyclic group $\mathbb{Z}$ naturally acts on $\mathbb{S}^3 \times \mathbb{R}$, by letting

$$m \cdot (z_1, z_2, t) = (z_1, z_2, t + m)$$

for $m \in \mathbb{Z}$; since $\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$, the quotient under this action is obviously isomorphic to the product $\mathbb{S}^3 \times \mathbb{S}^1$. The diffeomorphism φ allows us to transfer the action of $\mathbb{Z}$ on $\mathbb{S}^3 \times \mathbb{R}$ to an action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$. Explicitly, it is given by the formula

$$m \cdot (z_1, z_2) = \big(e^m z_1, e^m z_2\big).$$

The formula shows that $\mathbb{Z}$ acts by biholomorphisms; moreover, the action is clearly properly discontinuous and without fixed points. By Proposition 4.9, the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the action of $\mathbb{Z}$ is a complex manifold M. By construction, it is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$, and hence compact.

With the help of the Künneth formula from algebraic topology, we can compute the cohomology of the product $\mathbb{S}^3 \times \mathbb{S}^1$, and hence that of M. The result is that

$$b_0 = b_1 = b_3 = b_4 = 1, \quad b_2 = 0,$$

where $b_k = \dim H^k(M, \mathbb{R})$. It follows that M cannot possibly admit a Kähler metric, because ω would then define a nonzero class in $H^2(M, \mathbb{R})$, contradicting the fact that $b_2 = 0$. (Moreover, b_1 and b_3 are not even numbers.)

Complex projective space. An important example of a compact Kähler manifold is complex projective space $\mathbb{P}^n$. Its cohomology is easy to compute, using some results from algebraic topology

Lemma 19.2. *The cohomology groups of complex projective space are*

$$H^k(\mathbb{P}^n, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \textit{for } 0 \le k \le 2n \textit{ even,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. To save space, we omit the coefficients from cohomology groups. We prove the assertion by induction on $n \geq 0$, the case $n = 0$ being trivial (since $\mathbb{P}^0$ is a single point). Let $[z_0, z_1, \dots, z_n]$ be homogeneous coordinates on $\mathbb{P}^n$, and define $Z \subseteq \mathbb{P}^n$ as the set of points with $z_n = 0$. Clearly $Z \simeq \mathbb{P}^{n-1}$, and the complement $\mathbb{P}^n \setminus Z$ is isomorphic to $\mathbb{C}^n$, whose homology groups in positive degrees are zero. The Poincaré duality isomorphism

$$H^k(\mathbb{P}^n, Z) \simeq H_{2n-k}(\mathbb{P}^n \setminus Z) \simeq H_{2n-k}(\mathbb{C}^n)$$

now shows that $H^k(\mathbb{P}^n, Z) \simeq 0$ for $k < 2n$, while $H^{2n}(\mathbb{P}^n, Z) \simeq \mathbb{Z}$. We can then use the long exact cohomology sequence for the pair $(\mathbb{P}^n, Z)$,

$$\cdots \to H^k(\mathbb{P}^n, Z) \to H^k(\mathbb{P}^n) \to H^k(Z) \to H^{k+1}(\mathbb{P}^n, Z) \to \cdots$$

to conclude that the restriction map $H^k(\mathbb{P}^n) \to H^k(Z)$ is an isomorphism for $k \leq 2n-2$, and that $H^{2n-1}(\mathbb{P}^n) \simeq 0$. Likewise, we have

$$\cdots \to H^{2n-1}(Z) \to H^{2n}(\mathbb{P}^n, Z) \to H^{2n}(\mathbb{P}^n) \to H^{2n}(Z) \to \cdots,$$

and the terms at both ends are zero since $2n - 1 > 2 \dim Z = 2n - 2$. □

Recall that the Fubini-Study metric on $\mathbb{P}^n$ is Kähler, with Kähler form ω_{FS}. We have already seen that each $L^k(1) = \omega_{FS}^{\wedge k}$ is harmonic and gives a nonzero class in $H^{2k}(\mathbb{P}^n, \mathbb{R})$. Since this class is clearly of type (k,k), we conclude that

$$H^{p,q}(\mathbb{P}^n) \simeq \begin{cases} \mathbb{C} & \text{for } 0 \le p = q \le n, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the Hodge diamond of $\mathbb{P}^n$ has the following shape:

$$\begin{array}{ccccccccc}
 & & & & \mathbb{C} & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & \mathbb{C} & & 0 & & \\
 & \iddots & & & \vdots & & & \ddots & \\
0 & & 0 & & \vdots & & 0 & & 0 \\
 & \ddots & & & \vdots & & & \iddots & \\
 & & 0 & & \mathbb{C} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & \mathbb{C} & & & &
\end{array}$$

Complex tori. Another useful class of example are complex tori. Recall that a complex torus is a quotient of $\mathbb{C}^n$ by a lattice Λ, that is, a discrete subgroup isomorphic to $\mathbb{Z}^{2n}$. We have seen that $T = \mathbb{C}^n/\Lambda$ is a compact complex manifold (since the action of Λ by translation is properly discontinuous and without fixed points). The quotient map $\pi \colon \mathbb{C}^n \to T$ is locally biholomorphic, and so we can use small open subsets of $\mathbb{C}^n$ as coordinate charts on T. With this choice of coordinates, it is easy to see that the pullback map $\pi^* \colon A^{p,q}(T) \to A^{p,q}(\mathbb{C}^n)$ is injective and identifies $A^{p,q}(T)$ with the space of smooth (p,q)-forms on $\mathbb{C}^n$ that are invariant under translation by elements of Λ.

In fact, T has a natural Kähler metric: On $\mathbb{C}^n$, we have the Euclidean metric with Kähler form $\frac{i}{2}\sum dz_j \wedge d\bar{z}_j$, where $z_1, \dots, z_n$ are the coordinate functions on $\mathbb{C}^n$. This metric is invariant under translations, and thus descends to a Hermitian metric h on T. Let ω be the associated $(1,1)$-form; since $q^*\omega = \frac{i}{2}\sum dz_j \wedge d\bar{z}_j$, it is clear that $d\omega = 0$, and so h is a Kähler metric.

Lemma 19.3. *The Laplace operator for this metric is given by the formula*

$$\Delta\left(\sum \varphi_{J,K} dz_J \wedge d\bar{z}_K\right) = \sum \Delta\varphi_{J,K} \cdot dz_J \wedge d\bar{z}_K,$$

where $\Delta\varphi = -\sum_{j=1}^n \left(\partial^2\varphi/\partial x_j^2 + \partial^2\varphi/\partial y_j^2\right)$ *is the ordinary Laplacian on smooth functions.*

Proof. The injectivity of $\pi^* \colon A^{p,q}(T) \to A^{p,q}(\mathbb{C}^n)$ allows us to do the calculation on $\mathbb{C}^n$, where the metric is the standard one. In the notation from the appendix, we have

$$\Delta = 2\overline{\Box} = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) = 2\sum_{j,k=1}^n \left(\bar{\partial}_j \bar{e}_j \bar{e}_k^* \bar{\partial}_k^* + \bar{e}_k^* \bar{\partial}_k^* \bar{\partial}_j \bar{e}_j\right).$$

Now $\bar{\partial}_k^* = -\partial_k$, and so the summation simplifies to

$$-\sum_{j,k=1}^n \left(\bar{\partial}_j \bar{e}_j \bar{e}_k^* \partial_k + \bar{e}_k^* \partial_k \bar{\partial}_j \bar{e}_j\right) = -\sum_{j,k=1}^n \partial_k \bar{\partial}_j \left(\bar{e}_j \bar{e}_k^* + \bar{e}_k^* \bar{e}_j\right) = -2\sum_{j=1}^n \partial_j \bar{\partial}_j.$$

This means that we have

$$\Delta\left(\sum_{J,K} \varphi_{J,K} dz_J \wedge d\bar{z}_K\right) = -4\sum_{J,K}\sum_{j=1}^n \frac{\partial^2 \varphi_{J,K}}{\partial z_j \partial \bar{z}_j} dz_J \wedge d\bar{z}_K,$$

which gives the asserted formula because $4\partial^2\varphi/\partial z_j\partial\bar{z}_j = \partial^2\varphi/\partial x_j^2 + \partial^2\varphi/\partial y_j^2$. □

The lemma shows that the space $\mathcal{H}^0(T)$ of real-valued smooth functions on T that are harmonic for the metric h can be identified with the space of harmonic functions on $\mathbb{C}^n$ that are Λ-periodic. Since T is compact, we know that $\mathcal{H}^0(T) \simeq H^0(T,\mathbb{R}) \simeq \mathbb{R}$, and so any such function is constant. This means that all harmonic forms of type (p,q) on T can be described as

$$\sum_{|J|=p}\sum_{|K|=q} a_{J,K} dz_J \wedge d\bar{z}_K, \tag{19.4}$$

with constants $a_{J,K} \in \mathbb{C}$. Thus if we let $V_{\mathbb{R}} = H^1(T,\mathbb{R})$, then $H^1(T,\mathbb{C}) = V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, with $V^{1,0}$ generated by $dz_1, \dots, dz_n$, and $V^{0,1}$ by their conjugates.

Since any harmonic form as in (19.4) is a wedge product of forms in $V_{\mathbb{C}}$, it follows from the Hodge theorem that we have

$$H^k(T,\mathbb{C}) \simeq \bigwedge^k V_{\mathbb{C}},$$

and under this isomorphism, the Hodge decomposition of T is nothing but the abstract decomposition

$$\bigwedge^k V = \bigoplus_{p+q=k} V^{p,q}$$

into the subspaces $V^{p,q} = \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$. A basis for the space $V^{p,q}$ is given by the forms $dz_J \wedge d\bar{z}_K$ with $|J| = p$ and $|K| = q$. Note that we have $\dim V^{1,0} = \dim V^{0,1} = n$, and hence

$$h^{p,q} = \dim V^{p,q} = \binom{n}{p}\binom{n}{q}.$$

Example 19.5. Let T be a three-dimensional complex torus. Then the Hodge diamond of T has the following shape:

$$\begin{array}{ccccccc}
 & & & \mathbb{C} & & & \\
 & & \mathbb{C}^3 & & \mathbb{C}^3 & & \\
 & \mathbb{C}^3 & & \mathbb{C}^9 & & \mathbb{C}^3 & \\
\mathbb{C} & & \mathbb{C}^{10} & & \mathbb{C}^{10} & & \mathbb{C} \\
 & \mathbb{C}^3 & & \mathbb{C}^9 & & \mathbb{C}^3 & \\
 & & \mathbb{C}^3 & & \mathbb{C}^3 & & \\
 & & & \mathbb{C} & & &
\end{array}$$

Hypersurfaces in projective space. As a more involved (and more useful) example, we shall describe how to compute the Hodge numbers of a hypersurface in projective space. As usual, let $[z_0, z_1, \dots, z_{n+1}]$ denote the homogeneous coordinates on $\mathbb{P}^{n+1}$. Then any homogeneous polynomial $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ defines an analytic subset $Z(F)$, consisting of all points where $F(z) = 0$. (Different polynomials can define the same analytic set; but if we assume that F is not divisible by the square of any nonunit, then the zero set uniquely determines F by the Nullstellensatz from algebraic geometry.) If for every $z \neq 0$, at least one of the partial derivatives $\partial F/\partial z_j$ is nonzero, then $Z(F)$ is a complex submanifold of $\mathbb{P}^{n+1}$ of dimension n by the implicit mapping theorem (stated above as Theorem 7.3).

Note. We will show later that, in fact, any complex submanifold of projective space is defined by polynomial equations; moreover, if $M \subseteq \mathbb{P}^{n+1}$ has dimension n, then $M = Z(F)$ for a homogeneous polynomial $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$.

From now on, we fix $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ with the above properties, and let $M = Z(F)$ be the corresponding submanifold of $\mathbb{P}^{n+1}$. We also let $d = \deg F$ be the degree of the hypersurface. As usual, we give M the Kähler metric induced from the Fubini-Study metric on $\mathbb{P}^{n+1}$; then ω is the restriction of ω_{FS}. Since we know that the cohomology of $\mathbb{P}^{n+1}$ is generated by powers of ω_{FS}, and since the powers of ω define nonzero cohomology classes on M, the restriction map

$$H^k(\mathbb{P}^{n+1}, \mathbb{C}) \to H^k(M, \mathbb{C})$$

must be injective for $0 \leq k \leq 2n$. Now it is a fact (which we might prove later on) that the map is an isomorphism for $0 \leq k < n$. This result is known as the *Lefschetz hyperplane section theorem*; it implies that the cohomology of M is isomorphic to that of projective space in all degrees except $k = n$. In the remaining case, we have

$$H^n(M, \mathbb{C}) = H^n(\mathbb{P}^{n+1}, \mathbb{C}) \oplus H^n_0(M, \mathbb{C}),$$

where $H^n_0(M, \mathbb{C})$ is the so-called *primitive cohomology* of the hypersurface M. Note that the first summand, $H^n(\mathbb{P}^{n+1}, \mathbb{C})$, will be either one-dimensional (if n is even), or zero (if n is odd).

Griffiths' formula. The Hodge decomposition theorem shows that we have

$$H^n_0(M, \mathbb{C}) = H^{n,0}_0 \oplus H^{n-1,1}_0 \oplus \cdots \oplus H^{0,n}_0,$$

and a pretty result by Phillip Griffiths makes it possible to compute the dimensions of the various summands.

Theorem 19.6. *Let $M \subseteq \mathbb{P}^{n+1}$ be a complex submanifold of dimension n, defined by a homogeneous polynomial $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ of degree d. Then*

$$H^{p,n-p}_0 \simeq \frac{A^{n+1}(M, n+1-p)}{A^{n+1}(M, n-p) + dA^n(M, n-p)}, \tag{19.7}$$

where $A^k(M, \ell)$ denotes the space of rational k-forms on $\mathbb{P}^{n+1}$ with a pole of order at most ℓ along the hypersurface M, and d is the exterior derivative.

To explain Griffiths' formula, we recall that a rational $(n+1)$-form on $\mathbb{C}^{n+1}$ is an expression

$$\frac{A(z_1, \dots, z_{n+1})}{B(z_1, \dots, z_{n+1})} dz_1 \wedge \cdots \wedge dz_{n+1},$$

where $A, B \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ are polynomials, with B not identically zero. On the set of points where $B \neq 0$, this defines a holomorphic differential form, but there may be poles along the zero set of B. If we homogenize the expression (by replacing z_j with z_j/z_0 and multiplying through by a power of z_0), we see that rational $(n+1)$-forms on $\mathbb{P}^{n+1}$ can be described as

$$\frac{P(z_0, z_1, \dots, z_{n+1})}{Q(z_0, z_1, \dots, z_{n+1})} \Omega;$$

here Ω is given by the formula

$$\Omega = \sum_{j=0}^{n+1} (-1)^j z_j dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1},$$

and $P, Q \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ are homogeneous polynomials with $\deg P + (n+2) = \deg Q$. If the rational form has a pole of order at most ℓ along the hypersurface M, and no other poles, then we must have $Q = F^\ell$, and so $\deg P = \ell d - (n+2)$.

Likewise, one can prove by homogenizing rational n-forms on $\mathbb{C}^{n+1}$ that any rational n-form on $\mathbb{P}^{n+1}$ with a pole of order at most ℓ along M can be put into the form

$$\alpha = \sum_{0 \leq j < k \leq n+1} (-1)^{j+k} \frac{z_k P_j - z_j P_k}{F^\ell} dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1},$$

for homogeneous polynomials $P_j \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ of degree $\deg P_j = \ell d - (n+1)$. A short computation shows that we have

$$d\alpha = \frac{F \sum_j \frac{\partial P_j}{\partial z_j} - \ell \sum_j P_j \frac{\partial F}{\partial z_j}}{F^{\ell+1}} \Omega. \tag{19.8}$$

Returning to Griffiths' formula (19.7), every rational $(n+1)$-form with a pole of order at most $(n+1-p)$ along M can thus be written as $P\Omega/F^{n+1-p}$, with $P \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ homogeneous of degree $(n+1-p)d-(n+2)$. Writing S for the polynomial ring, and S_ℓ for the space of homogeneous polynomials of degree ℓ, we can say that

$$A^{n+1}(M, n+1-p) \simeq S_{(n+1-p)d-(n+2)},$$

by identifying the rational form $P\Omega/F^{n+1-p}$ with the homogeneous polynomial P. The formula in (19.8) shows that we have

$$A^{n+1}(M, n-p) + dA^n(M, n-p) \simeq \sum_{j=0}^{n+1} S_{(n-p)d-(n+1)} \frac{\partial F}{\partial z_j} + S_{(n-p)d-(n+2)} F.$$

The *Jacobian ideal* of the hypersurface M is the homogeneous ideal $J(F) \subseteq S$ generated by the partial derivatives of F,

$$J(F) = S\left(\frac{\partial F}{\partial z_0}, \frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_{n+1}}\right).$$

Recall that we always have $F \in J(F)$; this follows from the identity

$$G = \frac{1}{\deg G} \sum_{j=0}^{n+1} z_j \frac{\partial G}{\partial z_j}$$

for homogeneous polynomials G. With the help of the graded ring $R(F) = S/J(F)$, we can now restate Griffiths' formula for the Hodge decomposition of the primitive cohomology groups of M as follows: Suppose that $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ is an irreducible homogeneous polynomial of degree d, whose zero set $M = Z(F)$ is a submanifold of $\mathbb{P}^{n+1}$. Then the summands of the Hodge decomposition of the primitive cohomology of M can be described as

$$H_0^{p,n-p}(M) \simeq R(F)_{(n+1-p)d-(n+2)}. \tag{19.9}$$

To see this formula in action, let us compute a few examples:

Example 19.10. Let $M \subseteq \mathbb{P}^2$ be a smooth plane curve of degree d, defined by a homogeneous equation $F \in \mathbb{C}[z_0, z_1, z_2]$ with $\deg F = d$. Since $H^1(\mathbb{P}^2, \mathbb{C}) = 0$, we have $H_0^1(M) \simeq H^1(M, \mathbb{C})$ in this case. Griffiths' formula (with $n = 1$) says that

$$H^{1,0}(M) \simeq R(F)_{d-3} \simeq S_{d-3},$$

and so we find that the genus of the Riemann surface M is given by the formula

$$g = h^{1,0}(M) = \dim S_{d-3} = \binom{d-1}{2}.$$

So for instance, smooth plane cubic curves always have genus one.

Appendix: The standard proof of the Kähler identities

For people who are not afraid of big computations, here is the standard proof of the Kähler identities in Theorem 15.3. The two identities are conjugates of each other, which means that we only need to prove one of them. Moreover, the identities only involve the metric h and its first derivatives, and so they hold on a general Kähler manifold as soon as they are known on $\mathbb{C}^n$ with the Euclidean metric. It is therefore enough to prove the identity

$$[\Lambda, \partial] = i\bar{\partial}^*$$

on $\mathbb{C}^n$ with the Euclidean metric h. In this metric, dz_j is orthogonal to $d\bar{z}_k$, and to dz_k for $k \neq j$, while

$$h(dz_j, dz_j) = h(dx_j + idy_j, dx_j + idy_j) = g(dx_j, dx_j) + g(dy_j, dy_j) = 2.$$

More generally, we have $h(dz_J \wedge d\bar{z}_K, dz_J \wedge d\bar{z}_K) = 2^{|J|+|K|}$.

To facilitate the computation, we introduce a few additional but more basic operators on the spaces $A^{p,q} = A^{p,q}(\mathbb{C}^n)$. First, define

$$e_j \colon A^{p,q} \to A^{p+1,q}, \quad \alpha \mapsto dz_j \wedge \alpha$$

as well as its conjugate

$$\bar{e}_j \colon A^{p,q} \to A^{p,q+1}, \quad \alpha \mapsto d\bar{z}_j \wedge \alpha.$$

We then have

$$L\alpha = \omega \wedge \alpha = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \wedge \alpha = \frac{i}{2} \sum_{j=1}^{n} e_j \bar{e}_j \alpha.$$

Using the induced Hermitian inner product on forms, we then define the adjoint

$$e_j^* \colon A^{p,q} \to A^{p-1,q}$$

by the pointwise condition that $h(e_j\alpha, \beta) = h(\alpha, e_j^*\beta)$.

Lemma 19.11. *The adjoint e_j^* has the following properties:*

(1) If $j \notin J$, then $e_j^*(dz_J \wedge d\bar{z}_K) = 0$, while $e_j^*(dz_j \wedge dz_J \wedge d\bar{z}_K) = 2dz_J \wedge d\bar{z}_K$.
(2) $e_k e_j^* + e_j^* e_k = 2\,\mathrm{id}$ in case $j = k$, and 0 otherwise.

Proof. By definition, we have

$$h(e_j^* dz_J \wedge d\bar{z}_K, dz_L \wedge d\bar{z}_M) = h(dz_J \wedge d\bar{z}_K, dz_j \wedge dz_L \wedge d\bar{z}_M),$$

and since dz_j occurs only in the second term, the inner product is always zero, proving that $e_j^* dz_J \wedge d\bar{z}_K = 0$. On the other hand,

$$\begin{aligned} h(e_j^* dz_j \wedge dz_J \wedge d\bar{z}_K, dz_L \wedge d\bar{z}_M) &= h(dz_j \wedge dz_J \wedge d\bar{z}_K, dz_j \wedge dz_L \wedge d\bar{z}_M) \\ &= 2h(dz_J \wedge d\bar{z}_K, dz_L \wedge d\bar{z}_M), \end{aligned}$$

which is nonzero exactly when $J = L$ and $K = M$. From this identity, it follows that $e_j^* dz_j \wedge dz_J \wedge d\bar{z}_K = 2dz_J \wedge d\bar{z}_K$, establishing (1).

To prove (2) for $j = k$, observe that since $dz_j \wedge dz_j = 0$, we have

$$e_j^* e_j \big(dz_J \wedge d\bar{z}_K\big) = \begin{cases} 0 & \text{if } j \in J, \\ 2dz_J \wedge d\bar{z}_K & \text{if } j \notin J, \end{cases}$$

while

$$e_j e_j^*\big(dz_J \wedge d\bar{z}_K\big) = \begin{cases} 2dz_J \wedge d\bar{z}_K & \text{if } j \in J, \\ 0 & \text{if } j \notin J. \end{cases}$$

Taken together, this shows that $e_j e_j^* + e_j^* e_j = 2\,\mathrm{id}$. Finally, let us prove that $e_k e_j^* + e_j^* e_k = 0$ when $j \neq k$. By (1), this is clearly true on $dz_J \wedge d\bar{z}_K$ in case $j \notin J$. On the other hand,

$$e_k e_j^*\big(dz_j \wedge dz_J \wedge d\bar{z}_K\big) = 2e_k\big(dz_J \wedge d\bar{z}_K\big) = 2dz_k \wedge dz_J \wedge d\bar{z}_K$$

and

$$e_j^* e_k\big(dz_j \wedge dz_J \wedge d\bar{z}_K\big) = e_j^*\big(dz_k \wedge dz_j \wedge dz_J \wedge d\bar{z}_K\big) = -2dz_k \wedge dz_J \wedge d\bar{z}_K,$$

and the combination of the two proves the asserted identity. □

We also define the differential operator

$$\partial_j \colon A^{p,q} \to A^{p,q}, \quad \sum_{J,K} \varphi_{J,K} dz_J \wedge d\bar{z}_K \mapsto \sum_{J,K} \frac{\partial \varphi_{J,K}}{\partial z_j} dz_J \wedge d\bar{z}_K$$

and its conjugate

$$\bar{\partial}_j \colon A^{p,q} \to A^{p,q}, \quad \sum_{J,K} \varphi_{J,K} dz_J \wedge d\bar{z}_K \mapsto \sum_{J,K} \frac{\partial \varphi_{J,K}}{\partial \bar{z}_j} dz_J \wedge d\bar{z}_K.$$

Clearly, both commute with the operators e_j and e_j^*, as well as with each other. As before, let $\partial_j^* = -*\bar{\partial}_j*$ be the adjoint of ∂_j, and $\bar{\partial}_j^* = -*\partial_j*$ that of $\bar{\partial}_j$. Integration by parts (against compactly supported forms) proves the following lemma.

Lemma 19.12. *We have $\partial_j^* = -\bar{\partial}_j$ and $\bar{\partial}_j^* = -\partial_j$.*

We now turn to the proof of the crucial identity $[\Lambda, \partial] = i\bar{\partial}^*$.

Proof. All the operators in the identity can be expressed in terms of the basic ones, as follows. Firstly, $L = \frac{i}{2}\sum e_j \bar{e}_j$, and so the adjoint is given by the formula $\Lambda = -\frac{i}{2}\sum \bar{e}_j^* e_j^*$. Quite evidently, we have $\partial = \sum \partial_j e_j$ and $\bar{\partial} = \sum \bar{\partial}_j \bar{e}_j$, and after taking adjoints, we find that $\partial^* = -\sum \bar{\partial}_j e_j^*$ and that $\bar{\partial}^* = -\sum \partial_j \bar{e}_j^*$. Using these expressions, we compute that

$$\Lambda\partial - \partial\Lambda = -\frac{i}{2}\sum_{j,k}\Big(\bar{e}_j^* e_j^* \partial_k e_k - \partial_k e_k \bar{e}_j^* e_j^*\Big) = -\frac{i}{2}\sum_{j,k}\partial_k\Big(\bar{e}_j^* e_j^* e_k - e_k \bar{e}_j^* e_j^*\Big).$$

Now $\bar{e}_j^* e_j^* e_k - e_k \bar{e}_j^* e_j^* = \bar{e}_j^*(e_j^* e_k + e_k e_j^*)$, which equals $2\bar{e}_j^*$ in case $j = k$, and is zero otherwise. We conclude that

$$\Lambda\partial - \partial\Lambda = -i\sum_j \partial_j \bar{e}_j^* = i\bar{\partial}^*,$$

which is the Kähler identity we were after. □

One can use a similar computation to prove the identity

$$[L, \Lambda] = (p - n)\,\mathrm{id} \quad \text{on } A^p(M),$$

from Proposition 15.5, which we used to show that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ acts on the space of differential forms on M.

Proof. The identity involves no derivatives of the metric, and it is therefore sufficient to prove it for the Euclidean metric on $\mathbb{C}^n$. We shall use the operators e_j and e_j^* introduced on the previous two pages. As shown there, $L = \frac{i}{2} \sum e_j \bar{e}_j$ and $\Lambda = \frac{i}{2} \sum e_j^* \bar{e}_j^*$, and so we have

$$L\Lambda - \Lambda L = -\frac{1}{4} \sum_{j,k=1}^{n} \Bigl(e_k \bar{e}_k e_j^* \bar{e}_j^* - e_j^* \bar{e}_j^* e_k \bar{e}_k \Bigr) = \frac{1}{4} \sum_{j,k=1}^{n} \Bigl(e_k e_j^* \bar{e}_k \bar{e}_j^* - e_j^* e_k \bar{e}_j^* \bar{e}_k \Bigr)$$

For $j = k$, we can use the identity $e_j e_j^* + e_j^* e_j = 2 \,\mathrm{id}$ to compute that

$$\begin{aligned} \sum_{j=1}^{n} \Bigl(e_j e_j^* \bar{e}_j \bar{e}_j^* - e_j^* e_j \bar{e}_j^* \bar{e}_j \Bigr) &= \sum_{j=1}^{n} \Bigl(e_j e_j^* \bar{e}_j \bar{e}_j^* - (2 \,\mathrm{id} - e_j e_j^*)(2 \,\mathrm{id} - \bar{e}_j \bar{e}_j^*) \Bigr) \\ &= 2 \sum_{j=1}^{n} \Bigl(e_j e_j^* + \bar{e}_j \bar{e}_j^* - 2 \,\mathrm{id} \Bigr). \end{aligned}$$

On the other hand, we have $e_j e_k^* + e_k^* e_j = 0$ if $j \neq k$, and therefore

$$\sum_{j=1}^{n} \Bigl(e_k e_j^* \bar{e}_k \bar{e}_j^* - e_j^* e_k \bar{e}_j^* \bar{e}_k \Bigr) = \sum_{j=1}^{n} \Bigl(e_k e_j^* \bar{e}_k \bar{e}_j^* - e_k e_j^* \bar{e}_k \bar{e}_j^* \Bigr) = 0.$$

Combining the two individual calculations, we find that

$$L\Lambda - \Lambda L = \frac{1}{2} \sum_{j=1}^{n} \Bigl(e_j e_j^* + \bar{e}_j \bar{e}_j^* - 2 \,\mathrm{id} \Bigr) = \frac{1}{2} \sum_{j=1}^{n} \Bigl(e_j e_j^* + \bar{e}_j \bar{e}_j^* \Bigr) - n \,\mathrm{id}\,.$$

Now $e_j e_j^*$ acts as multiplication by 2 on $dz_J \wedge d\bar{z}_K$ whenever $j \in J$, and otherwise it is zero; the same is true for $\bar{e}_j \bar{e}_j^*$. Consequently, the operator $[L, \Lambda]$ multiplies $dz_J \wedge d\bar{z}_K$ by the integer $|J| + |K| - n$, as asserted. □