Class 18. The Hodge decomposition (October 31)

Consequences of the Kähler identities. The Kähler identities lead to many wonderful relations between the different operators that we have introduced; here we shall give the three most important ones.

Corollary 18.1. *On a Kähler manifold, the various Laplace operators are related to each other by the formula* $\overline{\Box} = \Box = \frac{1}{2}\Delta$.

Proof. By definition,

$$\Delta = dd^* + d^*d = (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial).$$

According to the Kähler identities in Theorem 15.3, we have $\bar\partial^* = i\partial\Lambda - i\Lambda\partial$, and therefore

$$\begin{aligned}\Delta &= (\partial + \bar\partial)(\partial^* - i\Lambda\partial + i\partial\Lambda) + (\partial^* - i\Lambda\partial + i\partial\Lambda)(\partial + \bar\partial)\\ &= \partial\partial^* + \bar\partial\partial^* - i\bar\partial\Lambda\partial + i\bar\partial\partial\Lambda + \partial^*\partial + \partial^*\bar\partial - i\Lambda\partial\bar\partial + i\partial\Lambda\bar\partial.\end{aligned}$$

Now $\partial^*\bar\partial = i[\Lambda, \bar\partial]\bar\partial = -i\big(\Lambda\bar\partial - \bar\partial\Lambda\big)\bar\partial = i\bar\partial\Lambda\bar\partial = -\partial^*\bar\partial$ by the other Kähler identity. The above formula consequently therefore simplifies to

$$\begin{aligned}\Delta &= \Box - i\bar\partial\Lambda\partial + i\bar\partial\partial\Lambda - i\Lambda\partial\bar\partial + i\partial\Lambda\bar\partial = \Box - i\bar\partial\Lambda\partial - i\partial\bar\partial\Lambda + i\Lambda\bar\partial\partial + i\partial\Lambda\bar\partial\\ &= \Box + i\partial(\Lambda\bar\partial - \bar\partial\Lambda) + i(\Lambda\bar\partial - \bar\partial\Lambda)\partial = \Box + \partial\partial^* + \partial^*\partial = 2\Box.\end{aligned}$$

The other formula, $\Delta = 2\overline{\Box}$, follows from this by conjugation. ∎

Corollary 18.2. *On a Kähler manifold, the Laplace operator* Δ *commutes with the operators* $*$, L, *and* Λ, *and satisfies* $\Delta A^{p,q}(M) \subseteq A^{p,q}(M)$. *In particular,* $*$, L, *and* Λ *preserve harmonic forms.*

Proof. By taking adjoints, we obtain from the second identity in Theorem 15.3 that

$$-i\bar\partial = (i\bar\partial^*)^* = [\Lambda, \partial]^* = [\partial^*, L] = \partial^* L - L\partial^*.$$

Using the resulting formula $L\partial^* = \partial^* L + i\bar\partial$, we compute that

$$\begin{aligned}L\Box &= L\partial\partial^* + L\partial^*\partial = \partial L\partial^* + \partial^* L\partial + i\bar\partial\partial\\ &= \partial\partial^* L + i\partial\bar\partial + \partial^*\partial L + i\bar\partial\partial = \partial\partial^* L + \partial^*\partial L = \Box L.\end{aligned}$$

Therefore $[\Delta, L] = 2[\Box, L] = 0$; after taking adjoints, we also have $[\Lambda, \Delta] = 0$. That Δ commutes with $*$ was shown in the homework; finally, $\Delta = 2\overline{\Box}$, and the latter clearly preserves the space $A^{p,q}(M)$. ∎

A nice consequence is that the Kähler form ω, which is naturally defined by the metric, is a harmonic form. Note that this is equivalent to the Kähler condition, since harmonic forms are always closed.

Corollary 18.3. *On a Kähler manifold, the Kähler form* ω *is harmonic.*

Proof. The constant function 1 is clearly harmonic; since $\omega = L(1)$, and since the operator L preserves harmonic functions, it follows that ω is harmonic. ∎

The Hodge decomposition. Now let M be a *compact* Kähler manifold, with Kähler form ω. We have seen in Corollary 18.1 that $\Delta = 2\overline{\Box}$; this implies that the Laplace operator Δ preserves the type of a form, and that a form is harmonic if and only if it is $\bar{\partial}$-harmonic. In particular, it follows that if a form $\alpha \in A^k(M)$ is harmonic, then its components $\alpha^{p,q} \in A^{p,q}(M)$ are also harmonic. Indeed, we have

$$0 = \Delta\alpha = \sum_{p+q=k} \Delta\alpha^{p,q},$$

and since each $\Delta\alpha^{p,q}$ belongs again to $A^{p,q}(M)$, we see that $\Delta\alpha^{p,q} = 0$.

Corollary 18.4. *On a compact Kähler manifold M, the space of harmonic forms decomposes by type as*

$$\mathcal{H}^k(M) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M),$$

where $\mathcal{H}^{p,q}(M)$ is the space of (p,q)-forms that are $\bar{\partial}$-harmonic (and hence also harmonic).

Since we know that each cohomology class contains a unique harmonic representative, we now obtain the famous *Hodge decomposition* of the de Rham cohomology of a compact Kähler manifold. We state it in a way that is independent of the particula Kähler metric.

Theorem 18.5. *Let M be a compact Kähler manifold. Then the de Rham cohomology with complex coefficients admits a direct sum decomposition*

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}, \tag{18.6}$$

with $H^{p,q}$ equal to the subset of those cohomology classes that contain a d-closed form of type (p,q). We have $H^{q,p} = \overline{H^{p,q}}$, where complex conjugation is with respect to the real structure on $H^k(M,\mathbb{C}) \simeq H^k(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$; moreover, $H^{p,q}$ is isomorphic to the Dolbeault cohomology group $H^{p,q}(M) \simeq H^q(M,\Omega_M^p)$.

Proof. Since M is a Kähler manifold, it admits a Kähler metric h, and we can consider forms that are harmonic for this metric. By Theorem 13.7, every class in $H^k(M,\mathbb{C})$ contains a unique complex-valued harmonic form α. Since $\alpha = \sum_{p+q=k} \alpha^{p,q}$, with each $\alpha^{p,q}$ harmonic and hence in $H^{p,q}$, we obtain the asserted decomposition. Note that by its very description, the decomposition does not depend on the choice of Kähler metric. Since the conjugate of a (p,q)-form is a (q,p)-form, it is clear that $\overline{H^{p,q}} = H^{q,p}$. Finally, every harmonic form is automatically $\bar{\partial}$-harmonic, and so we have $H^{p,q} \simeq \mathcal{H}^{p,q}(M) \simeq H^{p,q}(M)$ by Theorem 14.3. □

Recall the definition of the sheaf Ω_M^p holomorphic p-forms: its sections are smooth $(p,0)$-forms that can be expressed in local coordinates as

$$\alpha = \sum_{j_1 < \dots < j_k} f_{j_1,\dots,j_k} dz_{j_1} \wedge \dots \wedge dz_{j_k},$$

with locally defined holomorphic functions $f_{j_1,\dots,j_k}$. This expression shows that $\bar{\partial}\alpha = 0$. A useful (and surprising) fact is that on a compact Kähler manifold, any global holomorphic p-form is harmonic, and hence satisfies $d\alpha = 0$.

Corollary 18.7. *On a compact Kähler manifold M, every holomorphic form is harmonic, and so we get an embedding $H^0(M, \Omega_M^p) \hookrightarrow H^p(M, \mathbb{C})$ whose image is precisely the space $H^{p,0}$.*

Proof. If $\alpha \in A^{p,0}(M)$ is holomorphic, it satisfies $\bar{\partial}\alpha = 0$; on the other hand, $\bar{\partial}^*\alpha = 0$ since it would belong to the space $A^{p,-1}(M)$. Thus α is $\bar{\partial}$-harmonic, and hence also harmonic. □

The decomposition of the cohomology groups of M can be represented by the following picture, often called the *Hodge diamond* due to its shape.

$$
\begin{array}{ccccccc}
 & & & H^{n,n} & & & \\
 & & H^{n,n-1} & & H^{n-1,n} & & \\
 & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} & \\
 \cdots & & & & & & \cdots \\
H^{n,0} & H^{n-1,1} & \cdots & \cdots & \cdots & H^{1,n-1} & H^{0,n} \\
 \cdots & & & & & & \cdots \\
 & H^{2,0} & & H^{1,1} & & H^{0,2} & \\
 & & H^{1,0} & & H^{0,1} & & \\
 & & & H^{0,0} & & &
\end{array}
$$

It has several symmetries: On the one hand, we have $H^{q,p} = \overline{H^{p,q}}$; on the other hand, the $*$-operator induces an isomorphism between $H^{p,q}$ and $H^{n-q,n-p}$.

Example 18.8. Let M be a compact Riemannan surface. Then any Hermitian metric h on M is Kähler, and so we get the decomposition

$$H^1(M, \mathbb{C}) = H^{1,0} \oplus H^{0,1},$$

with $H^{1,0} \simeq H^0(M, \Omega_M^1)$ and $H^{0,1} \simeq H^1(M, \mathscr{O}_M)$. In particular, the dimension is $\dim H^1(M, \mathbb{R}) = 2g$, where $g = \dim_{\mathbb{C}} H^0(M, \Omega_M^1)$ is the *genus*. This means that the genus is a topological invariant of M, a fact that should be familiar from the theory of Riemann surfaces.

Example 18.9. Let us consider the case of a compact connected Kähler manifold of dimension two (so $n = 2$). In that case, the Hodge diamond looks like this:

$$
\begin{array}{ccccc}
 & & H^{2,2} & & \\
 & H^{2,1} & & H^{1,2} & \\
H^{2,0} & & H^{1,1} & & H^{0,2} \\
 & H^{1,0} & & H^{0,1} & \\
 & & H^{0,0} & &
\end{array}
$$

If we let $h^{p,q} = \dim H^{p,q}(M)$, then $h^{0,0} = h^{2,2} = 1$ since M is connected. Moreover, the two symmetries mentioned above show that $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2}$ and that

$h^{2,0} = h^{0,2}$. We also have $h^{1,1} \geq 1$, since the class of the Kähler form ω is a nonzero element of $H^{1,1}$.

Consequences of the Hodge decomposition. The Hodge decomposition theorem shows that compact Kähler manifold have various topological properties not shared by arbitrary complex manifolds.

Corollary 18.10. *On a compact Kähler manifold, the odd Betti numbers $b_{2k+1} = \dim H^{2k+1}(M, \mathbb{R})$ are always even.*

Proof. Indeed, $b_{2k+1} = \dim_{\mathbb{C}} H^{2k+1}(M, \mathbb{C})$. The latter decomposes as

$$H^{2k+1}(M, \mathbb{C}) = \bigoplus_{p+q=2k+1} H^{p,q},$$

and since $\dim_{\mathbb{C}} H^{p,q} = \dim_{\mathbb{C}} H^{q,p}$, we get the assertion. $\square$

Corollary 18.11. *On a compact Kähler manifold, the even Betti numbers b_{2k} are always nonzero.*

Proof. Since the operator $L = \omega \wedge (-)$ preserves harmonic forms, each $\omega^{\wedge k} = L^k(1)$ is harmonic; moreover, it is not zero because of Wirtinger's formula $\operatorname{vol}(M) = \frac{1}{n!} \int_M \omega^{\wedge n}$. Its cohomology class gives a nonzero element in $H^{2k}(M, \mathbb{R})$. $\square$

Another property of compact Kähler manifolds that is used very often in complex geometry is the following $\partial\bar{\partial}$-Lemma.

Proposition 18.12. *Let M be a compact Kähler manifold, and let ϕ be a smooth form that is both ∂-closed and $\bar{\partial}$-closed. If ϕ is also either ∂-exact or $\bar{\partial}$-exact, then it can be written as $\phi = \partial\bar{\partial}\psi$.*

Proof. We shall suppose that $\phi = \bar{\partial}\alpha$. Let $\alpha = \beta + \Delta\gamma$ be the decomposition given by (13.6), with β harmonic. We then have $2\bar{\square}\beta = \Delta\beta = 0$, and therefore $\bar{\partial}\beta = 0$. Using the previously mentioned identity $\bar{\partial}\partial^* = -\partial^*\bar{\partial}$, we compute that

$$\phi = \bar{\partial}\alpha = \bar{\partial}(2\square)\gamma = 2\bar{\partial}(\partial\partial^* + \partial^*\partial)\gamma = -2\partial\bar{\partial}(\partial^*\gamma) - 2\partial^*\bar{\partial}\partial\gamma.$$

Now $\partial\phi = 0$, and so the form $\partial^*\bar{\partial}\partial\gamma$ belongs to $\ker \partial \cap \operatorname{im} \partial^* = \{0\}$. Consequently, we have $\omega = \partial\bar{\partial}\psi$ with $\psi = -2\partial^*\gamma$. $\square$

The Lefschetz decomposition in cohomology. We showed earlier that the three operators

$$L(\alpha) = \omega \wedge \alpha, \quad \Lambda(\alpha) = (-1)^{\deg \alpha} * L * \alpha, \quad H(\alpha) = \big(\deg \alpha - n\big)\alpha$$

determine a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on the space of all forms

$$A^{n+*}(M) = \bigoplus_{k=-n}^{n} A^{n+k}(M).$$

We have just seen that all three operators actually commute with the Laplace operator Δ, and this makes the space of all *harmonic* forms

$$\mathcal{H}^{n+*}(M) = \bigoplus_{k=-n}^{n} \mathcal{H}^{n+k}(M)$$

is a subrepresentation. On a compact Kähler manifold, we can use the fact that every class in $H^k(M,\mathbb{C})$ is uniquely represented by a harmonic form to get an induced representation of $\mathfrak{sl}_2(\mathbb{C})$ on the total cohomology

$$H^{n+*}(M,\mathbb{R}) = \bigoplus_{k=-n}^{n} H^{n+k}(M,\mathbb{R}).$$

Let us denote the three operators in this representation by the symbols L_{coh}, Λ_{coh}, and H_{coh}, to emphasize that we are now looking at cohomology. To be precise, if $[\alpha] \in H^{n+k}(M,\mathbb{R})$ is a cohomology class, then

$$L_{coh}[\alpha] = [L\alpha_0], \quad \Lambda_{coh}[\alpha] = [\Lambda\alpha_0], \quad H_{coh}[\alpha] = [H\alpha_0],$$

where $\alpha_0 \in \mathcal{H}^k(M)$ is the harmonic representative of the given class. Of course,

$$H[\alpha] = (\deg\alpha - n)[\alpha] \quad \text{and} \quad L_{coh}[\alpha] = [\omega \wedge \alpha] = [\omega] \wedge [\alpha],$$

since the difference $\alpha - \alpha_0$ is exact. But $\Lambda_{coh}[\alpha] \neq [\Lambda\alpha]$; in fact, the latter does not even make sense usually, because Λ does not take closed forms to closed forms. (The commutator $[\Lambda, d]$ is not zero!)

From the $\mathfrak{sl}_2(\mathbb{C})$-representation, we obtain the following *Lefschetz decomposition* of the cohomology of M.

Theorem 18.13. *Let M be a compact Kähler manifold with Kähler form ω. Then every cohomology class $a \in H^{n+k}(M,\mathbb{C})$ admits a unique decomposition*

$$a = \sum_{j=\max(k,0)} \frac{L_{coh}^j}{j!} a_j,$$

with $a_j \in H^{n+k-2j}(M,\mathbb{R})$ primitive. This means that $\Lambda_{coh} a_j = 0$, or equivalently, that $L_{coh}^{2j-k+1} a_j = 0$.

The decomposition is compatible with the Hodge decomposition: ω is a $(1,1)$-form, and so $L_{coh} H^{p,q} \subseteq H^{p+1,q+1}$ and $\Lambda_{coh} H^{p,q} \subseteq H^{p-1,q-1}$, because this is true on the level of harmonic forms. It follows that if $a \in H^{p,q}$, then we get $a_j \in H^{p-j,q-j}$ for the components in the Lefschetz decomposition.

The best-known consequence of the Lefschetz decomposition is the following result, usually called the *Hard Lefschetz Theorem*, "hard" in the sense of "difficult".

Corollary 18.14. *The operator $L_{coh}^k \colon H^{n-k}(M,\mathbb{R}) \to H^{n+k}(M,\mathbb{R})$ is an isomorphism for every $k \geq 1$.*

Proof. This holds on the level of harmonic forms because of Corollary 16.8. □

The Hodge-Riemann bilinear relations. The last step in our proof of the Kähler identities was Weil's identity

$$w(\alpha) = \varepsilon(\alpha) J(*\alpha).$$

I already explained last time that this identity is very useful for describing the inner product $(\alpha,\beta)_M = \int_M \alpha \wedge *\overline{\beta}$ on the space of forms more in terms of representation theory. We can turn it around to describe the positivity of the natural pairing

$$H^k(M,\mathbb{R}) \otimes H^{2n-k}(M,\mathbb{R}) \to \mathbb{R}, \quad \big([\alpha],[\beta]\big) \mapsto \int_M \alpha \wedge \beta$$

on cohomology. First, we can use the Lefschetz isomorphism to turn this into a (nondegenerate) pairing on $H^k(M, \mathbb{R})$. Fix an integer $0 \leq k \leq n$, and define a bilinear form on the space $A^k(M)$ by the formula

$$Q(\alpha, \beta) = (-1)^{k(k-1)/2} \cdot (L^{n-k}\alpha, \beta)_M = (-1)^{k(k-1)/2} \int_M \omega^{n-k} \wedge \alpha \wedge \beta.$$

It is easy to see that $Q(\beta, \alpha) = (-1)^k Q(\alpha, \beta)$, and so Q is either linear or antilinear, depending on the parity of k. Moreover, if $d\alpha = d\beta = 0$, then the value of $Q(\alpha, \beta)$ only depends on the cohomology classes of α and β, and so this defines a pairing

$$Q \colon H^k(M, \mathbb{R}) \otimes H^k(M, \mathbb{R}) \to \mathbb{R}.$$

We obtain the so-called *Hodge-Riemann bilinear relations.*

Theorem 18.15. *The bilinear form $Q(\alpha, \beta) = (-1)^{k(k-1)/2} \int_M \omega^{n-k} \wedge \alpha \wedge \beta$ has the following two properties:*

(1) In the Hodge decomposition of $H^k(M, \mathbb{C})$, the subspaces $H^{p,q}$ and $H^{p',q'}$ are orthogonal to each other unless $p = q'$ and $q = p'$.
(2) For any nonzero primitive class $\alpha \in H^{p,q}$, we have $i^{p-q} Q(\alpha, \overline{\alpha}) > 0$.

Proof. Because $Q(\alpha, \beta)$ only depends on the cohomology classes of α and β, it is enough to consider the case where $\alpha, \beta \in \mathcal{H}^k(M)$ are harmonic forms. The first assertion is easy to see by looking at types. For the second one, suppose that $\alpha, \beta \in \mathcal{H}^{p,q}(M)$ are primitive with $p + q = k$, so that $\Lambda\alpha = \Lambda\beta = 0$. From Weil's identity (in Proposition 17.3), we get

$$(\alpha, \beta)_M = \int_M \alpha \wedge *\overline{\beta} = \frac{(-1)^{k(k-1)/2} i^{p-q}}{(n-k)!} \int_M \alpha \wedge L^{n-k}\overline{\beta} = \frac{i^{p-q}}{(n-k)!} Q(\alpha, \overline{\beta}).$$

This shows that $i^{p-q} Q(\alpha, \overline{\beta}) = (n-k)! \cdot (\alpha, \beta)_M$ is indeed a positive-definite inner product on the space $H^{p,q}$. □