Class 17. Proof of the Kähler identities (Oct 29)

Lefschetz decomposition and primitive forms. Today, we are finally going to prove the Kähler identities. Before we do that, let us first recall the general results about $\mathfrak{sl}_2(\mathbb{C})$-representations from last time in the case of differential forms. As usual, M is a Kähler manifold of dimension n, and L, Λ, and H are the three operators defined by

$$L(\alpha) = \omega \wedge \alpha, \quad \Lambda(\alpha) = (-1)^{\deg \alpha} * L * \alpha, \quad H(\alpha) = \big(\deg \alpha - n\big)\alpha.$$

They determine a representation of $\mathfrak{sl}_2(\mathbb{C})$ on the vector space

$$A^{n+*}(M) = \bigoplus_{k=-n}^{n} A^{n+k}(M).$$

We proved that every $\alpha \in A^{n+k}(M)$ has a unique Lefschetz decomposition

$$\alpha = \sum_{j \geq \max(0,k)} \frac{L^j}{j!} \alpha_j,$$

where $\alpha_j \in A^{n+k-2j}(M)$ is primitive, meaning that $\Lambda \alpha_j = 0$. We deduced that

$$L^k \colon A^{n-k}(M) \to A^{n+k}(M)$$

is an isomorphism for every $k \geq 1$. This leads to the following description of primitive forms in terms of L.

Lemma 17.1. *Let $k \geq 0$. A form $\alpha \in A^{n-k}(M)$ is primitive iff $L^{k+1}(\alpha) = 0$.*

Proof. If $\Lambda\alpha = 0$, then we get $L^{k+1}\alpha = 0$ from Lemma 16.1. It is therefore enough to prove the converse. Consider a form $\alpha \in A^{n-k}(M)$ such that $L^{k+1}\alpha = 0$. Since $k \leq 0$, the Lefschetz decomposition of α takes the form

$$\alpha = \sum_{j \geq 0} \frac{L^j}{j!} \alpha_j,$$

and so we get

$$0 = L^{k+1}\alpha = \sum_{j \geq 1} \frac{L^{j+k+1}}{j!} \alpha_j.$$

Since $j + k + 1 \leq k + 2j$ for $j \geq 1$, the uniqueness of the Lefschetz decomposition implies that $\alpha_j = 0$ for $j \geq 1$, which gives us that $\alpha = \alpha_0$ is primitive. □

We also introduced the Weyl element $w = \exp(L)\exp(-\Lambda)\exp(L)$, and noted that it induces an isomorphism between $A^{n-k}(M)$ and $A^{n+k}(M)$ for every $k \in \mathbb{Z}$. If $\alpha \in A^{n-k}(M)$ is primitive, then

$$w(\alpha) = \frac{L^k}{k!}\alpha;$$

in general, we can describe the action by w using the Lefschetz decomposition:

$$w(\alpha) = \sum_{j \geq \max(0,k)} (-1)^j \frac{L^{k-j}}{(k-j)!} \alpha_j.$$

Proof of the Kähler identities. Last time, we observed that $[H, d] = d$ and $[L, d] = 0$, due to the fact that $d\omega = 0$. We already deduced from this with the help of representation theory that

$$[\Lambda, d] = -w d w^{-1}. \tag{17.2}$$

Now let us see how this implies the Kähler identities $[\Lambda, d] = *i(\bar\partial - \partial)*$. The key ingredient is *Weil's identity*.

Proposition 17.3. *One has $w(\alpha) = \varepsilon(\deg \alpha) \cdot J(*\alpha)$ for every $\alpha \in A^k(X)$.*

Here $\varepsilon(k) = (-1)^{k(k-1)/2}$ and $J\colon A^k(M) \to A^k(M)$ is the (real) operator

$$J(\alpha) = \sum_{p+q=k} i^{p-q} \alpha^{p,q}.$$

Concretely, J acts by multiplying each occurence of dz_j by i, and each occurence of $d\bar z_j$ by $-i$; obviously, J is compatible with taking wedge products. It is easy to see that J is a real operator: if $\alpha \in A^k(M)$ is real, then $\overline{\alpha^{p,q}} = \alpha^{q,p}$, and then

$$\overline{J(\alpha)} = \sum_{p+q=k} \overline{i^{p-q}\alpha^{p,q}} = \sum_{p+q=k} i^{q-p}\alpha^{q,p} = J(\alpha)$$

shows that $J(\alpha) \in A^k(M)$ is again real. Morevoer, J and $*$ commute, because $\alpha \in A^{p,q}(M)$ implies that $*\alpha \in A^{n-q,n-p}(M)$, and so $J(*\alpha) = *(J\alpha)$.

We can understand where the sign factor $\varepsilon(k) = (-1)^{k(k-1)/2}$ comes from by looking at the following example.

Example 17.4. Recall that in the standard metric on $\mathbb{C}^n$, the pointwise length squared of $dz_j = dx_j + i dy_j$ is equal to 2. This gives

$$(dz_1 \wedge \dots \wedge dz_n) \wedge *(d\bar z_1 \wedge \dots \wedge d\bar z_n) = 2^n dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n.$$

By comparing this with

$$\begin{aligned}(dz_1 \wedge \dots \wedge dz_n) \wedge (d\bar z_1 \wedge \dots \wedge d\bar z_n) &= (-1)^{n(n-1)/2} (dz_1 \wedge d\bar z_1) \wedge \dots \wedge (dz_n \wedge d\bar z_n) \\ &= \varepsilon(n)(-2i)^n \cdot dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n\end{aligned}$$

we find that

$$*(d\bar z_1 \wedge \dots \wedge d\bar z_n) = \varepsilon(n)(-i)^n \cdot d\bar z_1 \wedge \dots \wedge d\bar z_n.$$

So $\varepsilon(n)$ shows up because of how we defined the standard orientation on a complex manifold.

Here are two simple identities for $\varepsilon(k)$ that we will use below:

$$\begin{aligned}\varepsilon(k+1) &= (-1)^k \varepsilon(k) \\ \varepsilon(k+\ell) &= (-1)^{k\ell}\varepsilon(k)\varepsilon(\ell)\end{aligned}$$

Proof of the Kähler identities. Fix $\alpha \in A^k(M)$. We then have

$$*^2\alpha = (-1)^k \alpha, \quad J^2\alpha = (-1)^k\alpha, \quad w^2\alpha = (-1)^{k-n}\alpha,$$

the last formula being due to $H(\alpha) = (k-n)\alpha$. From (17.2), we get

$$[\Lambda, d]\,\alpha = -wdw^{-1}\,\alpha = -(-1)^{k-n} wdw\,\alpha = -(-1)^{k-n}\varepsilon(k)\varepsilon(2n+1-k) * JdJ * \alpha,$$

using Weil's identity and $[J, *] = 0$ in the last step. Now

$$JdJ\,\alpha = \sum_{p+q=k} i^{p-q}\left(i^{p+1-q}\partial\alpha + i^{p-q-1}\bar\partial\alpha\right) = (-1)^k (i\partial\alpha - i\bar\partial\alpha),$$

and so we can rewrite the formula above as

$$[\Lambda, d]\, \alpha = (-1)^n \varepsilon(k)\varepsilon(2n+1-k) * (i\bar{\partial} - i\partial) * \alpha.$$

This gives us the Kähler identity $[\Lambda, d] = *(i\bar{\partial} - i\partial)*$ if we observe that

$$(-1)^n \varepsilon(k)\varepsilon(2n+1-k) = (-1)^n (-1)^{k(2n+1-k)} \varepsilon(2n+1) = 1. \qquad \square$$

Proof of Weil's identity. All that is left to do is to prove Weil's identity. The advantage is that this is again a pointwise statement. Fix a point $p \in M$, and consider the $2n$-dimensional real vector space $V = T^*_{\mathbb{R},p}M$. In addition to the inner product $g = g_p$ and the element $\omega = \omega_p \in \bigwedge^2 V$ coming from the Kähler form, we now also have the operator $J = J_p$ acting on $\bigwedge V$. Recall that $T_{\mathbb{R},p}M$ is isomorphic to the holomorphic tangent space T'_pM. Choose local coordinates $z_1, \dots, z_n$ centered at p such that $\partial/\partial z_1, \dots, \partial/\partial z_n$ form an orthonormal basis in T'_pM. Then the $2n$ real vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial y_1, \dots, \partial/\partial y_n$ form an othornomal basis in $T_{\mathbb{R},p}M$, and in this basis, we have $J(\partial/\partial x_j) = \partial/\partial y_j$ and $J(\partial/\partial y_j) = -\partial/\partial x_j$. We use the (positively oriented) dual basis $dx_1, dy_1, \dots, dx_n, dy_n$ in $T^*_{\mathbb{R},p}M$. Since $J(dz_j) = i dz_j$ and $J(d\bar{z}_j) = -i d\bar{z}_j$, we have

$$J(dx_j) = \frac{J(dz_j) + J(d\bar{z}_j)}{2} = \frac{i dz_j - i d\bar{z}_j}{2} = -dy_j \quad \text{and} \quad J(dy_j) = dx_j.$$

In this way, we find an orthonormal basis $e_1, e_2, \dots, e_{2n-1}, e_{2n} \in V$ that is positively oriented and has the property that

$$\omega = e_1 \wedge e_2 + \cdots + e_{2n-1} \wedge e_{2n}$$

and that $J(e_1) = -e_2$ and $J(e_2) = e_1$ and so on. We can now prove Weil's identity by induction on $n \geq 1$.

Proof of Weil's identity. We again treat the case $n = 1$ first. Here $V = \mathbb{R}^2$, with basis e_1, e_2; the other basis elements are $1 \in \bigwedge^0 V$ and $e_1 \wedge e_2 \in \bigwedge^2 V$. We already know from the proof of Proposition 15.5 that

$$*(1) = e_1 \wedge e_2, \quad *(e_1) = e_2, \quad *(e_2) = -e_1, \quad *(e_1 \wedge e_2) = 1.$$

Together with $J(e_1) = -e_2$ and $J(e_2) = e_1$ and the fact that J is compatible with wedge products, this gives

$$J * (1) = e_1 \wedge e_2, \quad J * (e_1) = e_1, \quad J * (e_2) = e_1, \quad J * (e_1 \wedge e_2) = 1,$$

If we compare this against

$$w(1) = e_1 \wedge e_2, \quad w(e_1) = e_1, \quad w(e_2) = e_2, \quad w(e_1 \wedge e_2) = -1,$$

we see that the required sign changes are exactly $\varepsilon(0) = 1$, $\varepsilon(1) = 1$, and $\varepsilon(2) = -1$. Weil's identity therefore holds for $n = 1$.

In order to do the general case, we use induction on the dimension. If $n \geq 2$, we again decompose $V = V_1 \oplus V_2$, for example by letting V_1 be the span of $e_1, \dots, e_{2n-2}$, and letting V_2 be the span of e_{2n-1} and e_{2n}. By induction, we already know the identity on V_1 and V_2. So it is again enough to prove that Weil's identity is compatible with direct sums.

So let us consider more generally the case where $V = V_1 \oplus V_2$. Suppose that each $\dim V_j = 2n_j$ is even, and that we have a positively oriented orthonormal basis in each V_j; let us denote by $*_j$ the $*$-operator on V_j, and by $\omega_j \in \bigwedge^2 V_j$ the "Kähler form". We use the notation J_j and w_j for the other two operators that appear in Weil's identity; by induction on the dimension, we can assume that the identity holds on V_1 and V_2. The union of the two orthonormal bases, with the same ordering, gives us a positively oriented orthonormal basis on $V = V_1 \oplus V_2$. We keep the symbols $*$, w, and J for the resulting operators on V. During the proof of Proposition 15.5, we showed that the isomorphism

$$\bigwedge V \cong \bigwedge V_1 \otimes \bigwedge V_2$$

is actually an isomorphism of $\mathfrak{sl}_2(\mathbb{C})$-representations. Keeping the same notation, it follows that the Weyl element acts according to the rule

$$w(\alpha \wedge \beta) = (w_1\alpha) \wedge (w_2\beta).$$

We also know already that

$$*(\alpha \wedge \beta) = (-1)^{\deg \alpha \cdot \deg \beta} (*_1\alpha) \wedge (*_2\beta).$$

The rest of the proof is a simple computation:

$$\begin{aligned} w(\alpha \wedge \beta) = (w_1\alpha) \wedge (w_2\beta) &= \varepsilon(\alpha)\varepsilon(\beta) \cdot (J_1 *_1 \alpha) \wedge (J_2 *_2 \beta) \\ &= \varepsilon(\alpha)\varepsilon(\beta)(-1)^{\deg \alpha \cdot \deg \beta} \cdot J * (\alpha \wedge \beta) \\ &= \varepsilon(\deg \alpha + \deg \beta) \cdot J * (\alpha \wedge \beta) \end{aligned}$$

Because $\deg(\alpha \wedge \beta) = \deg \alpha + \deg \beta$, this gives us the result we want. □

A formula for the inner product. Now suppose that M is compact. Weil's identity also makes it possible to describe the inner product between two forms $\alpha, \beta \in A^{p,q}(M)$ using the Weyl element. Set $k = p + q$. Recall that

$$(\alpha, \beta)_M = \int_M \alpha \wedge *\bar{\beta}.$$

Since $\bar{\beta} \in A^{q,p}(M)$, Weil's identity becomes

$$w(\bar{\beta}) = \varepsilon(k) * (J\bar{\beta}) = \varepsilon(k) i^{q-p} * \bar{\beta}.$$

Substituting into the formula for the inner product, we get

$$(\alpha, \beta)_M = i^{p-q}\varepsilon(k) \cdot \int_M \alpha \wedge w(\bar{\beta})$$

So the $\mathfrak{sl}_2(\mathbb{C})$-representation actually gives us the inner product as well.