Class 13. Harmonic theory (October 10)

We can now return to the problem of finding canonical representatives for classes in $H^k(M, \mathbb{R})$ on a compact oriented Riemannian manifold (M, g). Following the general strategy outlined last time, we put inner products on the spaces of forms $A^k(M)$, and use these to define an adjoint d^* for the exterior derivative, and a Laplace operator $\Delta = dd^* + d^*d$.

Linear algebra. We begin by discussing some more linear algebra. Let V be a real vector space of dimension n, with inner product $g\colon V \times V \to \mathbb{R}$. (The example we have in mind is $V = T_{\mathbb{R},p}M$, with the inner product g_p coming from the Riemannian metric.) The inner product yields an isomorphism

$$\varepsilon\colon V \to V^*, \quad v \mapsto g(v, -),$$

between V and its dual space $V^* = \operatorname{Hom}(V, \mathbb{R})$. Note that if $e_1, \dots, e_n$ is an orthonormal basis for V, then $\varepsilon(e_1), \dots, \varepsilon(e_n)$ is the dual basis in V^*. We endow V^* with the inner product induced by the isomorphism ε, and then this dual basis becomes orthonormal as well.

All the spaces $\bigwedge^k V$ also acquire inner products, by setting

$$g(u_1 \wedge \dots \wedge u_k, v_1 \wedge \dots \wedge v_k) = \det\bigl(g(u_i, v_j)\bigr)_{i,j=1}^k$$

and extending bilinearly. These inner products have the property that, for any orthonormal basis $e_1, \dots, e_n \in V$, the vectors

$$e_{i_1} \wedge \dots \wedge e_{i_k}$$

with $i_1 < i_2 < \dots < i_k$ form an orthonormal basis for $\bigwedge^k V$.

Now suppose that V is in addition oriented. Recall that the fundamental element $\phi \in \bigwedge^n V$ is the unique positive vector of length 1; we have $\phi = e_1 \wedge \dots \wedge e_n$ for any positively-oriented orthonormal basis.

Definition 13.1. The *$*$-operator* is the unique linear operator $*\colon \bigwedge^k V \to \bigwedge^{n-k} V$ with the property that $\alpha \wedge *\beta = g(\alpha, \beta) \cdot \phi$ for any $\alpha, \beta \in \bigwedge^k V$.

Note that $\alpha \wedge *\beta$ belongs to $\bigwedge^n V$, and is therefore a multiple of the fundamental element ϕ. The $*$-operator is most conveniently defined using an orthormal basis $e_1, \dots, e_n$ for V: for any permutation σ of $\{1, \dots, n\}$, we have

$$e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(n)} = \operatorname{sgn}(\sigma) \cdot e_1 \wedge \dots \wedge e_n = \operatorname{sgn}(\sigma) \cdot \phi,$$

and consequently

$$* \left(e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(k)}\right) = \operatorname{sgn}(\sigma) \cdot e_{\sigma(k+1)} \wedge \dots \wedge e_{\sigma(n)}. \tag{13.2}$$

This relation shows that $*$ takes an orthonormal basis to an orthonormal basis, and is therefore an isometry: $g(*\alpha, *\beta) = g(\alpha, \beta)$.

Lemma 13.3. *We have $** \alpha = (-1)^{k(n-k)}\alpha$ for any $\alpha \in \bigwedge^k V$.*

Proof. Let $\alpha, \beta \in \bigwedge^k V$. By definition of the $*$-operator, we have

$$\begin{aligned}(** \alpha) \wedge (*\beta) &= (-1)^{k(n-k)}(*\beta) \wedge (** \alpha) = (-1)^{k(n-k)} g(*\beta, *\alpha) \cdot \phi \\ &= (-1)^{k(n-k)} g(\alpha, \beta) \cdot \phi = (-1)^{k(n-k)} \alpha \wedge *\beta.\end{aligned}$$

This being true for all β, we conclude that $** \alpha = (-1)^{k(n-k)}\alpha$. ∎

It follows that $*\colon \bigwedge^k V \to \bigwedge^{n-k} V$ is an isomorphism; this may be viewed as an abstract form of Poincaré duality (which says that on a compact oriented manifold, $H^k(M,\mathbb{R}) \simeq H^{n-k}(M,\mathbb{R})$ for every $0 \le k \le n$).

Inner products and the Laplacian. Let (M,g) be a Riemannian manifold that is compact, oriented, and of dimension n. At every point $p \in M$, we have an inner product g_p on the real tangent space $T_{\mathbb{R},p}M$, and therefore also on the cotangent space $T^*_{\mathbb{R},p}M$ and on each $\bigwedge^k T^*_{\mathbb{R},p}M$. In other words, each vector bundle $\bigwedge^k T^*_{\mathbb{R}}M$ carries a natural Euclidean metric. This allows us to define an inner product on the space of smooth k-forms $A^k(M)$ by the formula

$$(\alpha,\beta)_M = \int_M g\bigl(\alpha,\beta\bigr)\operatorname{vol}(g).$$

The individual $*$-operators $*\colon \bigwedge^k T^*_{\mathbb{R},p}M \to \bigwedge^{n-k} T^*_{\mathbb{R},p}M$ at each point $p \in M$ give us a a linear mapping

$$*\colon A^k(M) \to A^{n-k}(M).$$

By definition, we have $\alpha \wedge *\beta = g(\alpha,\beta)\cdot \operatorname{vol}(g)$, and so the inner product can also be expressed by the simpler formula

$$(\alpha,\beta)_M = \int_M \alpha \wedge *\beta.$$

It has the advantage of hiding the terms coming from the metric.

We already know that the exterior derivative d is a linear differential operator. Since the bundles in question carry Euclidean metrics, there is a unique adjoint; the $*$-operator allows us to write down a simple formula for it.

Proposition 13.4. *The adjoint $d^*\colon A^k(M) \to A^{k-1}(M)$ is given by the formula*

$$d^* = -(-1)^{n(k+1)} * d * .$$

Proof. Fix $\alpha \in A^{k-1}(M)$ and $\beta \in A^k(M)$. By Stokes' theorem, the integral of $d\bigl(\alpha \wedge *\beta\bigr) = d\alpha \wedge *\beta + (-1)^{k-1}\alpha \wedge d(*\beta)$ over M is zero, and therefore

$$(d\alpha,\beta)_M = \int_M d\alpha \wedge *\beta = (-1)^k \int_M \alpha \wedge d * \beta = (-1)^k \int_M \alpha \wedge *\bigl(*^{-1} d * \beta\bigr)$$

This shows that the adjoint is given by the formula $d^*\beta = (-1)^k *^{-1} d * \beta$. Since $d * \beta \in A^{n-k+1}(M)$, we can use the identity from Lemma 13.3 to compute that

$$d^*\beta = (-1)^k (-1)^{(n-k+1)(k-1)} * d * \beta,$$

from which the assertion follows because $k^2 + k$ is an even number. □

Definition 13.5. For each $0 \le k \le n$, we define the *Laplace operator* $\Delta\colon A^k(M) \to A^k(M)$ by the formula $\Delta = d \circ d^* + d^* \circ d$. A form $\omega \in A^k(M)$ is called *harmonic* if $\Delta\omega = 0$, and we let $\mathcal{H}^k(M) = \ker \Delta$ be the space of all harmonic forms.

More precisely, each Δ is a second-order linear differential operator from the vector bundle $\bigwedge^k T^*_{\mathbb{R}}M$ to itself. It is easy to see that Δ is formally self-adjoint; indeed, the adjointness of d and d^* shows that

$$(\Delta\alpha,\beta)_M = (d\alpha, d\beta)_M + (d^*\alpha, d^*\beta)_M = (\alpha, \Delta\beta)_M.$$

By computing a formula for Δ in local coordinates, one shows that Δ is an elliptic operator. We may therefore apply the fundamental theorem of elliptic operators

(Theorem 12.9) to conclude that the space of harmonic forms $\mathcal{H}^k(M)$ is finite-dimensional, and that we have an orthogonal decomposition

$$A^k(M) = \mathcal{H}^k(M) \oplus \operatorname{im}\big(\Delta\colon A^k(M) \to A^k(M)\big). \tag{13.6}$$

We can now state and prove the main theorem of real Hodge theory.

Theorem 13.7. *Let (M, g) be a compact and oriented Riemannian manifold. Then the natural map $\mathcal{H}^k(M) \to H^k(M, \mathbb{R})$ is an isomorphism; in other words, every de Rham cohomology class contains a unique harmonic form.*

Proof. Recall that a form ω is harmonic iff $d\omega = 0$ and $d^*\omega = 0$; this follows from the identity $(\Delta\omega, \omega)_M = \|d\omega\|_M^2 + \|d^*\omega\|_M^2$. In particular, harmonic forms are automatically closed, and therefore define classes in de Rham cohomology. We have to show that the resulting map $\mathcal{H}^k(M) \to H^k(M, \mathbb{R})$ is bijective.

To prove the injectivity, suppose that $\omega \in \mathcal{H}^k(M)$ is harmonic and d-exact, say $\omega = d\psi$ for some $\psi \in A^{k-1}(M)$. Then

$$\|\omega\|_M^2 = (\omega, d\psi)_M = (d^*\omega, \psi)_M = 0,$$

and therefore $\omega = 0$. Note that this part of the proof is elementary, and does not use any of the results from the theory of elliptic operators.

To prove the surjectivity, take an arbitrary cohomology class and represent it by some $\alpha \in A^k(M)$ with $d\alpha = 0$. The decomposition in (13.6) shows that we have

$$\alpha = \omega + \Delta\beta = \omega + dd^*\beta + d^*d\beta.$$

with $\omega \in \mathcal{H}^k(M)$ harmonic and $\beta \in A^k(M)$. Since $d\omega = 0$, we get $0 = d\alpha = dd^*d\beta$, and therefore

$$\|d^*d\beta\|_M^2 = (d^*d\beta, d^*d\beta)_M = (d\beta, dd^*d\beta)_M = 0,$$

proving that $d^*d\beta = 0$. This shows that $\alpha = \omega + dd^*\beta$, and so the harmonic form ω represents the original cohomology class. □

Note. The space of harmonic forms $\mathcal{H}^k(M)$ depends on the Riemannian metric g; this is because the definition of the operators d^* and Δ involves the metric.

More linear algebra. Our next goal is to extend the Hodge theorem to the Dolbeault cohomology groups $H^{p,q}(M)$ on a compact complex manifold M with a Hermitian metric h. Recall that this means a collection of positive definite Hermitian forms $h_p\colon T'_pM \times T'_pM \to \mathbb{C}$ on the holomorphic tangent spaces that vary smoothly with the point $p \in M$.

As in the case of Riemannian manifolds, we begin by looking at a single Hermitian vector space (V, h); in our applications, $V = T'_pM$ will be the holomorphic tangent space to a complex manifold. Thus let V be a complex vector space of dimension n, and $h\colon V \times V \to \mathbb{C}$ a positive definite form that is linear in its first argument, and satisfies $h(v_2, v_1) = \overline{h(v_1, v_2)}$.

We denote the underlying real vector space by $V_{\mathbb{R}}$, noting that it has dimension $2n$. Multiplication by i defines a linear operator $J \in \operatorname{End}(V_{\mathbb{R}})$ with the property that $J^2 = -\operatorname{id}$. The complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$ is a complex vector space of dimension $2n$; it decomposes into a direct sum

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}, \tag{13.8}$$

where $V^{1,0} = \ker(J - i\,\mathrm{id})$ and $V^{0,1} = \ker(J + i\,\mathrm{id})$ are the two eigenspaces of J. For any $v \in V_{\mathbb{R}}$, we have $v = \frac{1}{2}(v - iJv) + \frac{1}{2}(v + iJv)$; this means that the inclusion $V_{\mathbb{R}} \hookrightarrow V_{\mathbb{C}}$, followed by the projection $V_{\mathbb{C}} \twoheadrightarrow V^{1,0}$, defines an $\mathbb{R}$-linear map

$$V_{\mathbb{R}} \to V^{1,0}, \quad v \mapsto \frac{1}{2}(v - iJv)$$

which is an isomorphism of real vector spaces. This justifies identifying the original complex vector space V with the space $V^{1,0}$. (But we will see in a moment that, from the point of view of the inner product h, this is not quite the correct way to make the identification.)

The decomposition in (13.8) induces a decomposition

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \Bigl(\bigwedge^p V^{1,0}\Bigr) \otimes \Bigl(\bigwedge^q V^{0,1}\Bigr) = \bigoplus_{p+q=k} V^{p,q}, \tag{13.9}$$

and elements of $V^{p,q}$ are often said to be of *type* (p,q).

We have already seen that the Hermitian form h defines an inner product $g = \operatorname{Re} h$ on the real vector space $V_{\mathbb{R}}$. It satisfies $g(Jv_1, Jv_2) = g(v_1, v_2)$, and conversely, we can recover h from g by the formula

$$h(v_1, v_2) = g(v_1, v_2) + ig(v_1, Jv_2).$$

As usual, g induces inner products on the spaces $\bigwedge^k V_{\mathbb{R}}$, which we extend sesquilinearly to Hermitian inner products h on $\bigwedge^k V_{\mathbb{C}}$. We compute that

$$h(v_1 - iJv_1, v_2 - iJv_2) = 2\bigl(g(v_1, v_2) + ig(v_1, Jv_2)\bigr),$$

and this shows that the correct way to identify V with $V^{1,0}$ is via the isomorphism

$$V \to V^{1,0}, \quad v \mapsto \frac{1}{\sqrt{2}}\bigl(v - iJv\bigr),$$

because then the original Hermitian inner product h on V will agree with the induced Hermitian inner product on $V^{1,0}$.

Lemma 13.10. *The decomposition in* (13.9) *is orthogonal with respect to the Hermitian inner product h.*

Recall that $V_{\mathbb{R}}$ is automatically oriented; the natural orientation is given by $v_1, Jv_1, \dots, v_n, Jv_n$ for any complex basis $v_1, \dots, v_n \in V$. It follows that if $e_1, \dots, e_n$ is any orthonormal basis of V with respect to the Hermitian inner product h, then

$$e_1, Je_1, e_2, Je_2, \dots, e_n, Je_n$$

is a positively oriented orthonormal basis for $V_{\mathbb{R}}$; in particular, the fundamental element is given by the formula $\varphi = (e_1 \wedge Je_1) \wedge \dots \wedge (e_n \wedge Je_n)$.

As usual, we have the $*$-operator $\bigwedge^k V_{\mathbb{R}} \to \bigwedge^{2n-k} V_{\mathbb{R}}$; we extend it $\mathbb{C}$-linearly to $*\colon \bigwedge^k V_{\mathbb{C}} \to \bigwedge^{2n-k} V_{\mathbb{C}}$. Since we obtained the Hermitian inner product h on $\bigwedge^k V_{\mathbb{C}}$ by extending g linearly in the first and conjugate-linearly in the second argument, the $*$-operator satisfies the identity

$$\alpha \wedge *\overline{\beta} = h(\alpha, \beta) \cdot \varphi$$

for $\alpha, \beta \in \bigwedge^k V_{\mathbb{C}}$.

Lemma 13.11. *The $*$-operator maps $V^{p,q}$ into $V^{n-q,n-p}$, and satisfies $*^2\alpha = (-1)^{p+q}\alpha$ for any $\alpha \in V^{p,q}$.*

Proof. For $\beta \in V^{p,q}$ and $\alpha \in V^{r,s}$, we have $\alpha \wedge *\beta = h(\alpha, \overline{\beta}) \cdot \varphi = 0$ unless $p = s$ and $q = r$; this is because the decomposition by type is orthogonal (Lemma 13.10). It easily follows that $*\beta$ has type $(n-q, n-p)$. The second assertion is a restatement of Lemma 13.3, where we proved that $*^2 = (-1)^{k(2n-k)} \operatorname{id} = (-1)^k \operatorname{id}$ on $\bigwedge^k V_{\mathbb{R}}$. $\square$

The dual vector space $V_{\mathbb{R}}^* = \operatorname{Hom}(V_{\mathbb{R}}, \mathbb{R})$ also has a complex structure J, by defining $(Jf)(v) = f(Jv)$ for $f \in V_{\mathbb{R}}^*$ and $v \in V_{\mathbb{R}}$. Note that the isomorphism

$$\varepsilon \colon V_{\mathbb{R}} \to V_{\mathbb{R}}^*, \quad v \mapsto g(v, -)$$

is only conjugate-linear, since $\varepsilon(Jv) = g(Jv, -) = -g(v, J-) = -J\varepsilon(v)$.