Class 11. Other cohomology theories (October 3)

Example 11.1. Let us return to the exponential sequence on a complex manifold M. From Proposition 10.14, we obtain a long exact sequence

$$0 \longrightarrow H^0(M, \mathbb{Z}_M) \longrightarrow H^0(M, \mathscr{O}_M) \longrightarrow H^0(M, \mathscr{O}_M^*) \longrightarrow H^1(M, \mathbb{Z}_M) \longrightarrow \cdots$$

One can show that the cohomology groups $H^i(M, \mathbb{Z}_M)$ are (naturally) isomorphic to the singular cohomology groups $H^i(M, \mathbb{Z})$ defined in algebraic topology. Thus whether or not the map $\mathscr{O}_M(M) \to \mathscr{O}_M^*(M)$ is surjective depends on the group $H^1(M, \mathbb{Z})$; for instance, $H^1(\mathbb{C}^*, \mathbb{Z}) \simeq \mathbb{Z}$, and this explains the failure of surjectivity. On the other hand, if M is simply connected, then $H^1(M, \mathbb{Z}) = 0$, and therefore $\mathscr{O}_M(M) \to \mathscr{O}_M^*(M)$ is surjective.

Example 11.2. Let $0 \to \mathscr{F} \to \operatorname{ds} \mathscr{F} \to \mathscr{F} / \operatorname{ds} \mathscr{F} \to 0$ be the short exact sequence at the start of the Godement resolution. Because $\operatorname{ds} \mathscr{F}$ is flasque, it has no higher cohomology, and so

$$0 \to H^0(X, \mathscr{F}) \to H^0(X, \operatorname{ds} \mathscr{F}) \to H^0(X, \mathscr{F} / \operatorname{ds} \mathscr{F}) \to H^1(X, \mathscr{F}) \to 0$$

is exact. It is a fun exercise to think about what kind of description of $H^1(X, \mathscr{F})$ we get in this way.

Čech cohomology. In addition to the general framework introduced above, there are many other cohomology theories; one that is often convenient for calculations is Čech cohomology. We shall limit our discussion to a special case that will be useful later.

Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups. Fix an open cover $\mathbf{U}$ of X. The group of *p-cochains* for the cover is the product

$$C^p(\mathbf{U}, \mathscr{F}) = \prod_{U_0, \dots, U_p \in \mathbf{U}} \mathscr{F}(U_0 \cap U_1 \cap \dots \cap U_p);$$

we denote a typical element by $\mathbf{g}$, with components $g_{U_0, \dots, U_p} \in \mathscr{F}(U_0 \cap \dots \cap U_p)$. The restriction maps for the sheaf $\mathscr{F}$ allow us to define a *differential* $\delta^p \colon C^p(\mathbf{U}, \mathscr{F}) \to C^{p+1}(\mathbf{U}, \mathscr{F})$ by setting $\delta^p(\mathbf{g}) = \mathbf{h}$, where

$$h_{U_0, \dots, U_{p+1}} = \sum_{k=0}^{p+1} (-1)^k g_{U_0, \dots, U_{k-1}, U_{k+1}, \dots, U_{p+1}} |_{U_0 \cap U_1 \cap \dots \cap U_{p+1}}.$$

Then a somewhat tedious computation shows that $\delta^{p+1} \circ \delta^p = 0$, and thus

$$0 \longrightarrow C^0(\mathbf{U}, \mathscr{F}) \xrightarrow{\delta^0} C^1(\mathbf{U}, \mathscr{F}) \xrightarrow{\delta^1} C^2(\mathbf{U}, \mathscr{F}) \xrightarrow{\delta^2} \cdots \tag{11.3}$$

is a complex of abelian groups. We define the *Čech cohomology group* $H^i(\mathbf{U}, \mathscr{F})$ to be the i-th cohomology group of the complex.

Example 11.4. From the sheaf axioms, one readily proves that $H^0(\mathbf{U}, \mathscr{F}) \simeq \mathscr{F}(X)$.

Example 11.5. Let $L \to M$ be a holomorphic line bundle on a complex manifold M. The transition functions $g_{\alpha,\beta} \in \mathscr{O}_M^*(U_\alpha \cap U_\beta)$ satisfy the relations $g_{\alpha,\beta} \cdot g_{\beta,\gamma} = g_{\alpha,\gamma}$. In other words, we have a cohomology class in $H^1(\mathbf{U}, \mathscr{O}_M^*)$. If this class is trivial, we have $g_{\alpha,\beta} = s_\beta / s_\alpha$ for $s_\alpha \in \mathscr{O}_M^*(U_\alpha)$, which means that the s_α^{-1} form a nowhere vanishing section of the line bundle. Thus we can think of $H^1(\mathbf{U}, \mathscr{O}_M^*)$ as the obstruction to the existence of such a section.

One can define Čech cohomology groups more generally, but unless the topological space X is nice, they lack the good properties of Godement's theory (for instance, there is not in general a long exact cohomology sequence). This drawback notwithstanding, Čech cohomology can frequently be used to compute the groups $H^i(X, \mathscr{F})$. The following result, known as Cartan's lemma, is the main result in this direction.

Theorem 11.6. *Suppose that the cover* $\mathbf{U}$ *is* acyclic *for the sheaf* $\mathscr{F}$*, in the sense that* $H^i(U_1 \cap \cdots \cap U_p, \mathscr{F}) = 0$ *for every* $U_1, \dots, U_p \in \mathbf{U}$ *and every* $i > 0$*. Then there are natural isomorphisms*

$$H^i(\mathbf{U}, \mathscr{F}) \simeq H^i(X, \mathscr{F})$$

between the Čech cohomology and the usual cohomology of $\mathscr{F}$*.*

The proof is not that difficult, but we leave it out since it requires a knowledge of spectral sequences.

Example 11.7. Let $\mathbf{U} = \{U_0, U_1\}$ be the standard open cover of $\mathbb{P}^1$. A good excercise in the use of Čech cohomology is to prove that $H^0(\mathbf{U}, \mathscr{O}) = \mathbb{C}$, while $H^j(\mathbf{U}, \mathscr{O}) = 0$ for $j \geq 1$. We will show later today that this cover is acyclic, and therefore $H^j(\mathbb{P}^1, \mathscr{O}) = 0$ for $j \geq 1$. (The vanishing of $H^1(\mathbb{P}^1, \mathscr{O})$ is exactly the existence of Laurent series.)

Dolbeault cohomology. On a complex manifold M, there is another way to compute the cohomology groups of the sheaves $\mathscr{O}_M$ and Ω_M^p (and, more generally, of the sheaf of sections of any holomorphic vector bundle), by relating them to Dolbeault cohomology. Recall that we had defined the Dolbeault cohomology groups

$$H^{p,q}(M) = \frac{\ker \bar{\partial} \colon A^{p,q}(M) \to A^{p,q+1}(M)}{\operatorname{coker} \bar{\partial} \colon A^{p,q-1}(M) \to A^{p,q}(M)},$$

where $A^{p,q}(M)$ denotes the space of smooth (p,q)-forms on M. Clearly, each $H^{p,q}(M)$ is a complex vector space, and can also be viewed as the q-th cohomology group of the complex

$$0 \longrightarrow A^{p,0}(M) \xrightarrow{\bar{\partial}} A^{p,1}(M) \xrightarrow{\bar{\partial}} A^{p,2}(M) \longrightarrow \cdots \longrightarrow A^{p,n}(M) \longrightarrow 0.$$

The purpose of today's class is to prove the following result, usually referred to as Dolbeault's theorem. Recall that Ω_M^p is the sheaf of holomorphic p-forms; its sections over an open set $U \subseteq M$ are all smooth $(p,0)$-forms $\alpha \in A^{p,0}(U)$ such that $\bar{\partial}\alpha = 0$.

Theorem 11.8. *On a complex manifold* M*, we have natural isomorphisms*

$$H^q\big(M, \Omega_M^p\big) \simeq H^{p,q}(M)$$

for every $p, q \in \mathbb{N}$*.*

The proof is based on the $\bar{\partial}$-Poincaré lemma (Lemma 8.1) and some general sheaf theory. We fix an integer $p \geq 0$, and consider the complex of sheaves

$$0 \longrightarrow \Omega_M^p \longrightarrow \mathscr{A}^{p,0} \xrightarrow{\bar{\partial}} \mathscr{A}^{p,1} \xrightarrow{\bar{\partial}} \mathscr{A}^{p,2} \longrightarrow \cdots \longrightarrow \mathscr{A}^{p,n} \longrightarrow 0. \tag{11.9}$$

It is a complex because $\bar{\partial} \circ \bar{\partial} = 0$; the first observation is that it is actually exact.

Lemma 11.10. *The complex of sheaves in* (11.9) *is exact.*

Proof. It suffices to prove the exactness at the level of stalks; after fixing a point of M and choosing local coordinates, we may assume without loss of generality that M is an open subset of $\mathbb{C}^n$. Now let $\omega \in A^{p,q}(U)$ be defined on some open neighborhood of the point in question, and suppose that $\bar{\partial}\omega = 0$. If $q = 0$, this means that ω is holomorphic, and therefore $\omega \in \Omega_M^p(U)$, proving that the complex is exact at $\mathscr{A}^{p,0}$. If, on the other hand, $q > 0$, then Lemma 8.1 shows that there is a possibly smaller open neighborhood $V \subseteq U$ such that $\omega = \bar{\partial}\psi$ for some $\psi \in A^{p,q-1}(V)$, and so we have exactness on stalks. □

We will show in a moment that the higher cohomology groups for each of the sheaves $\mathscr{A}^{p,q}$ vanish. Assuming this for the time being, let us complete the proof of Theorem 11.8

Proof. Probably the most convenient way to get the conclusion is by using a spectral sequence; but since it is not difficult either, will shall give a more basic proof. We begin by breaking up (11.9) into several short exact sequences:

$$
\begin{tikzcd}
\Omega^p_M \arrow[dr] & & & & \mathscr{Q}^2 \arrow[dr] & & & & \mathscr{Q}^4 \arrow[dr] & \\
& \mathscr{A}^{p,0} \arrow[rr, "\bar{\partial}"] \arrow[dr] & & \mathscr{A}^{p,1} \arrow[rr, "\bar{\partial}"] \arrow[ur] & & \mathscr{A}^{p,2} \arrow[rr, "\bar{\partial}"] \arrow[dr] & & \mathscr{A}^{p,3} \arrow[rr] \arrow[ur] & & \cdots \\
& & \mathscr{Q}^1 \arrow[ur] & & & & \mathscr{Q}^3 \arrow[ur] & & &
\end{tikzcd}
\tag{11.11}
$$

Here $\mathscr{Q}^k = \ker\big(\bar{\partial}\colon \mathscr{A}^{p,k} \to \mathscr{A}^{p,k+1}\big) = \operatorname{im}\big(\bar{\partial}\colon \mathscr{A}^{p,k-1} \to \mathscr{A}^{p,k}\big)$, using that the original complex is exact.

Now recall that we have $H^0\big(M, \mathscr{A}^{p,q}\big) = A^{p,q}(M)$. Since $\mathscr{Q}^{q+1}$ is a subsheaf of $\mathscr{A}^{p,q+1}$, the sequence $0 \to \mathscr{Q}^q \to \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1}$ is exact. After passage to cohomology, we find that

$$\ker\big(\bar{\partial}\colon A^{p,q}(M) \to A^{p,q+1}(M)\big) \simeq H^0\big(M, \mathscr{Q}^q\big).$$

Also, $0 \to \mathscr{Q}^{q-1} \to \mathscr{A}^{p,q-1} \to \mathscr{Q}^q \to 0$ is exact, and as part of the corresponding long exact sequence, we have

$$A^{p,q-1}(M) \longrightarrow H^0\big(M, \mathscr{Q}^q\big) \longrightarrow H^1\big(M, \mathscr{Q}^{q-1}\big) \longrightarrow H^1\big(M, \mathscr{A}^{p,q}\big).$$

The fourth term vanishes, and we conclude that $H^{p,q}(M) \simeq H^1\big(M, \mathscr{Q}^{q-1}\big)$. Continuing in this manner, we then obtain a string of isomorphisms

$$H^{p,q}(M) \simeq H^1\big(M, \mathscr{Q}^{q-1}\big) \simeq H^2\big(M, \mathscr{Q}^{q-2}\big) \simeq \cdots \simeq H^{q-1}\big(M, \mathscr{Q}^1\big) \simeq H^q\big(M, \Omega_M^p\big),$$

which is the desired result. □

Applications. As an application of Dolbeault's theorem, we will now solve a classical problem about the geometry of $\mathbb{C}^n$. Let $X \subseteq \mathbb{C}^n$ be a *hypersurface*; this means that X is an analytic subset, locally defined by the vanishing of a single holomorphic function. We would like to show that, actually, $X = Z(f)$ for a global $f \in \mathscr{O}(\mathbb{C}^n)$.

This in another instance of a local-to-global problem, and we should expect the answer to come from cohomology. By assumption, X can locally be defined by a one holomorphic equation, and so we may cover $\mathbb{C}^n$ by open sets U_j, with the property that $X \cap U_j = Z(f_j)$ for certain $f_j \in \mathscr{O}(U_j)$; if an open set U_j does not meet X, we simply take $f_j = 1$. More precisely, we shall assume that each U_j is a polybox, that is, an open set of the form

$$\left\{ z \in \mathbb{C}^n \;\middle|\; |x_j - a_j| < r_j \text{ and } |y_j - b_j| < s_j \right\}.$$

Since the intersection of two open intervals is again an open interval, it is clear that every finite intersection of open sets in the cover $\mathbf{U}$ is again a polybox, and in particular contractible. Moreover, if we take the defining equation f_j not divisible by the square of any nonunit, then it is unique up to multiplication by units.

Next, we observe that if $D \subseteq \mathbb{C}^n$ is an arbitrary polybox, then $H^q(D, \Omega_D^p) = 0$ for $q > 0$; indeed, this group is isomorphic to $H^{p,q}(D)$, which vanishes for polyboxes by a result analogous to Proposition 8.5. In particular, the cover $\mathbf{U}$ is acyclic for the sheaf $\mathscr{O}$, and we have

$$H^q(\mathbf{U}, \mathscr{O}) \simeq H^q(\mathbb{C}^n, \mathscr{O}) \simeq H^{0,q}(\mathbb{C}^n) \simeq 0$$

by Cartan's lemma (Theorem 11.6) and Proposition 8.5.

Returning to the problem at hand, consider the intersection $U_j \cap U_k$. There, we have $f_j = g_{j,k} \cdot f_k$ for a nowhere vanishing holomorphic function $g_{j,k} \in \mathscr{O}^*(U_j \cap U_k)$. Now $U_j \cap U_k$ is contractible, and so $H^1(U_j \cap U_k, \mathbb{Z}) = 0$. From the exponential sequence

$$0 \longrightarrow \mathbb{Z}_{\mathbb{C}^n} \longrightarrow \mathscr{O}_{\mathbb{C}^n} \longrightarrow \mathscr{O}_{\mathbb{C}^n}^* \longrightarrow 0,$$

it follows that $g_{j,k} = e^{2\pi i h_{j,k}}$ for holomorphic functions $h_{j,k}$ on $U_j \cap U_k$. Observe that we have $g_{j,k} g_{k,l} = g_{j,l}$, and that $a_{j,k,l} = h_{j,l} - h_{j,k} - h_{k,l}$ is therefore an integer. These integers define a class in the Čech cohomology group

$$H^2(\mathbf{U}, \mathbb{Z}_{\mathbb{C}^n}) \simeq H^2(\mathbb{C}^n, \mathbb{Z}_{\mathbb{C}^n}) \simeq H^2(\mathbb{C}^n, \mathbb{Z}) \simeq 0.$$

The first isomorphism is because of Cartan's lemma (Theorem 11.6), since every intersection of open sets in the cover is contractible; the second and third isomorphisms are facts from algebraic topologyy. We thus have $a_{j,k,l} = b_{k,l} - b_{j,l} + b_{j,k}$ for integers $b_{j,k}$. Replacing $h_{j,k}$ by $h_{j,k} + b_{j,k}$, we may thus assume from the start that $h_{j,k} + h_{k,l} = h_{j,l}$ on $U_j \cap U_k \cap U_l$. This means that $\mathbf{h}$ defines an element of the Čech cohomology group $H^1(\mathbf{U}, \mathscr{O})$.

But as observed above, we have $H^1(\mathbf{U}, \mathscr{O}) \simeq 0$; this means that $h_{j,k} = h_k - h_j$ for holomorphic functions $h_j \in \mathscr{O}(U_j)$. This essentially completes the proof: By construction, $f_j = e^{2\pi i (h_k - h_j)} f_k$, and so $f_j e^{2\pi i h_j} = f_k e^{2\pi i h_k}$ on $U_j \cap U_k$. Since $\mathscr{O}$ is a sheaf, there is a holomorphic function $f \in \mathscr{O}(\mathbb{C}^n)$ with $f|_{U_j} = f_j e^{2\pi i h_j}$; clearly, we have $Z(f) = X$, proving that the hypersurface X is indeed defined by a single holomorphic equation.

Note. We proved the vanishing of the Dolbeault cohomology groups by purely analytic means in Proposition 8.5. To deduce from it the vanishing of Čech cohomology, we first go from Dolbeault cohomology to sheaf cohomology (Dolbeault's theorem), and then from sheaf cohomology to Čech cohomology (Cartan's lemma).

Fine and soft sheaves. We now have to explain why the higher cohomology groups of $\mathscr{A}^{p,q}$ vanish. This is due to the fact that sections of this sheaf are smooth forms, and that we have partitions of unity.

A few basic definitions first. An open covering $X = \bigcup_{i \in I} U_i$ of a topological space is *locally finite* if every point is contained in at most finitely many U_i. A topological space is called *paracompact* if every open cover can be refined to a locally finite open cover. It is not hard to see that a locally compact Hausdorff space with a countable basis is paracompact; in particular, every complex manifold is paracompact.

Definition 11.12. A sheaf $\mathscr{F}$ on a paracompact space X is *fine* if for every locally finite open cover $X = \bigcup_{i \in I} U_i$, there are sheaf homomorphisms $\eta_i \colon \mathscr{F} \to \mathscr{F}$, with the following two properties:

(1) There are open sets $V_i \supseteq X \setminus U_i$, such that $\eta_i \colon \mathscr{F}_x \to \mathscr{F}_x$ is the zero map for every $x \in V_i$.
(2) As morphisms of sheaves, $\sum_{i \in I} \eta_i = \mathrm{id}_{\mathscr{F}}$.

The first condition is saying that the support of $\eta_i(s)$ lies inside U_i; the second condition means that $s = \sum_{i \in I} \eta_i(s)$, which makes sense since the sum is locally finite. Note that if $s \in \mathscr{F}(U_i)$, then $\rho_i(s)$ may be considered as an element of $\mathscr{F}(X)$: by assumption, $\rho_i(s)$ is zero near the boundary of U_i, and can therefore be extended by zero using the sheaf axioms.

Example 11.13. On a complex manifold M, each $\mathscr{A}^{p,q}$ is a fine sheaf. Indeed, given any locally finite open covering $M = \bigcup_{i \in I} U_i$, we can find a partition of unity $1 = \sum_{i \in I} \rho_i$ subordinate to that cover; this means that each ρ_i is a smooth function with values in $[0, 1]$, and zero on an open neighborhood $V_i \supseteq M \setminus U_i$. We can now define $\eta_i \colon \mathscr{A}^{p,q} \to \mathscr{A}^{p,q}$ as multiplication by ρ_i; then both conditions in the definition are clearly satisfied.

Example 11.14. One can also show that the sheaf of discontinuous sections $\mathrm{ds}\,\mathscr{F}$ is always a fine sheaf.

We would like to show that fine sheaves have vanishing higher cohomology. But unfortunately, being fine does not propagate very well along the Godement resolution of a sheaf; this leads us to introduce a weaker property that does behave well in exact sequences of sheaves. We first observe that, just as in the case of geometric spaces, a sheaf $\mathscr{F}$ can be restricted to any closed subset $Z \subseteq X$; at each point $x \in Z$, the stalk of the restriction $\mathscr{F}|_Z$ is equal to $\mathscr{F}_x$. The precise definition is as follows: for $U \subseteq Z$, we let $\Gamma\big(U, \mathscr{F}|_Z\big)$ be the set of maps $s \colon U \to T(\mathscr{F})$ with $s(x) \in \mathscr{F}_x$ for every $x \in Z$, such that s is locally the restriction of a section of $\mathscr{F}$. (Here $T(\mathscr{F})$ is the disjoint union of all the stalks of $\mathscr{F}$.) We sometimes write $\mathscr{F}(Z)$ in place of the more correct $\Gamma(Z, \mathscr{F}|_Z)$.

Definition 11.15. A sheaf $\mathscr{F}$ on a paracompact topological space is called *soft* if, for every closed subset $Z \subseteq X$, the restriction map $\Gamma(X, \mathscr{F}) \to \Gamma(Z, \mathscr{F}|_Z)$ is surjective.

It is clear that the sheaf of discontinuous sections $\mathrm{ds}\,\mathscr{F}$ is soft for every sheaf $\mathscr{F}$. Let us now see why fine sheaves are soft. Fix an arbitrary section $t \in \Gamma(Z, \mathscr{F}|_Z)$; we need to show that it can be extended to a section of $\mathscr{F}$ on all of X. By definition, there certainly exist local extensions, and so we can find open sets $U_i \subseteq X$ whose union covers Z, and sections $s_i \in \Gamma(U_i, \mathscr{F})$ with $s_i(x) = t(x)$ for every $x \in Z$. We will assume that $U_0 = X \setminus Z$ is one of the open sets, with $s_0 = 0$. Since X is paracompact, we can assume after suitable refinement that the open cover of X by the U_i is locally finite; as $\mathscr{F}$ is fine, we can then find morphisms $\rho_i \colon \mathscr{F} \to \mathscr{F}$ as in Definition 11.12. After extending by zero, we may again consider $\rho_i(s_i) \in \mathscr{F}(X)$.

Now let

$$s = \sum_{i \in I} \rho_i(s_i) \in \Gamma(X, \mathscr{F}),$$

which makes sense since the sum is locally finite. For $x \in Z$, we have $s_i(x) = t(x)$ for every $i \neq 0$, and thus $s(x) = t(x)$. This proves the surjectivity of the map $\Gamma(X, \mathscr{F}) \to \Gamma(Z, \mathscr{F}|_Z)$, and shows that fine sheaves are soft.

Proposition 11.16. *Let $\mathscr{F}$ be a fine sheaf on a paracompact Hausdorff space X. Then $H^i(X, \mathscr{F}) = 0$ for every $i > 0$.*

We will show that the statement is true for the larger class of soft sheaves. The proof is very similar to that of Proposition 10.16; the first step is to study short exact sequences.

Lemma 11.17. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves on a paracompact space X, and if $\mathscr{F}'$ is soft, then $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0$ is an exact sequence of abelian groups.*

Proof. Again, we let $\alpha \colon \mathscr{F}' \to \mathscr{F}$ and $\beta \colon \mathscr{F} \to \mathscr{F}''$ denote the maps. By Lemma 10.9, it suffices to show that $\beta \colon \mathscr{F}(X) \to \mathscr{F}''(X)$ is surjective, and so we fix a global section $s'' \in \mathscr{F}''(X)$. The map being surjective locally, and X being paracompact, we can find a locally finite cover $X = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathscr{F}(U_i)$ such that $\beta(s_i) = s''|_{U_i}$. Now paracompact spaces are automatically normal, and so we can find closed sets $K_i \subseteq U_i$ whose interiors still cover X. Note that the union of any number of K_i is always closed; this is a straightforward consequence of the local finiteness of the cover.

We now consider the set of all pairs (K, s), where K is a union of certain K_i (and hence closed), and $s \in \Gamma(K)$ satisfies $\beta(s) = s''|_K$. As before, every chain has a maximal element, and so Zorn's lemma guarantees the existence of a maximal element (K_{max}, s_{max}). We claim that $K_{max} = X$; in other words, that $K_i \subseteq K_{max}$ for every $i \in I$. In any case, the two sections s_i and s_{max} both map to s'' on the intersection $K_i \cap K_{max}$, and we can therefore find $s' \in \mathscr{F}'(K_i \cap K_{max})$ with the property that $\alpha(s') = (s_{max} - s_i)|_{K_i \cap K_{max}}$. But $\mathscr{F}'$ is soft by assumption, and so there exists $t' \in \mathscr{F}'(K_i)$ with $t'|_{K_i \cap K_{max}} = s'$. Then s_{max} and $s_i + \alpha(t')$ agree on the overlap $K_i \cap K_{max}$, and thus define a section of $\mathscr{F}$ on $K_i \cup K_{max}$ lifting s''. By maximality, we have $K_i \cup K_{max} = K_{max}$, and hence $K_i \subseteq K_{max}$ as claimed. $\square$

Secondly, we need to know that the quotient of soft sheaves is soft.

Lemma 11.18. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence with $\mathscr{F}'$ and $\mathscr{F}$ soft, then $\mathscr{F}''$ is also soft.*

Proof. For any closed subset $Z \subseteq X$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{F}(X) & \longrightarrow & \mathscr{F}''(X) \\ \downarrow & & \downarrow \\ \mathscr{F}(Z) & \longrightarrow & \mathscr{F}''(Z). \end{array}$$

The surjectivity of the two horizontal maps is due to Lemma 11.17, and that of the vertical restriction map comes from the softness of $\mathscr{F}$. We conclude that $\mathscr{F}''(X) \to \mathscr{F}''(Z)$ is also surjective, proving that $\mathscr{F}''$ is soft. $\square$

We are now ready to prove Proposition 11.16.

Proof. According to the preceding lemma, the quotient of a soft sheaf by a soft subsheaf is again soft. This fact implies that in (10.12), all the the sheaves $\mathscr{G}^j$ are also soft sheaves. Consequently, the entire diagram remains exact after taking global sections, which shows that $0 \to \mathscr{F}(X) \to \mathscr{F}^0(X) \to \mathscr{F}^1(X) \to \cdots$ is an exact sequence of abelian groups. But this means that $H^i(X, \mathscr{F}) = 0$ for $i > 0$. $\square$

Since the sheaves $\mathscr{A}^{p,q}$ admit partitions of unity, they are fine, and hence soft. Proposition 11.16 now puts the last piece into place for the proof of Theorem 11.8.

Corollary 11.19. *On a complex manifold M, we have $H^i\big(M, \mathscr{A}^{p,q}\big) = 0$ for every $i > 0$.*

Note. Underlying the proof of Theorem 11.8 is a more general principle, which you should try to prove by yourself: If $0 \to \mathscr{F} \to \mathscr{E}^0 \to \mathscr{E}^1 \to \cdots$ is a resolution of $\mathscr{F}$ by acyclic sheaves (meaning that $H^i(X, \mathscr{E}^k) = 0$ for all $i > 0$), then the complex $0 \to \mathscr{E}^0(X) \to \mathscr{E}^1(X) \to \cdots$ computes the cohomology groups of $\mathscr{F}$. This can be seen either by breaking up the long exact sequence into short exact sequences as in (10.12), or by a spectral sequence argument.