Class 10. Sheaves and cohomology (October 1)

The Wirtinger theorem. Let (M, h) be a complex manifold with a Hermitian metric. Locally, there always exist unitary frames for the metric h, that is, smooth sections $\zeta_1, \dots, \zeta_n$ of $T'M$ whose values give a unitary basis for the holomorphic tangent space T'_pM at each point. For such a frame, we have

$$h(\zeta_j, \zeta_k) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

One way to construct such a unitary frame is to start from an arbitrary frame (for instance, the coordinate vector fields $\partial/\partial z_1, \dots, \partial/\partial z_n$), and then apply the Gram-Schmidt process. If we let $\theta_1, \dots, \theta_n$ be a dual basis of smooth $(1,0)$-forms, in the sense that $\theta_j(\zeta_k) = 1$ if $j = k$, and 0 otherwise, then we have

$$\omega = \frac{i}{2} \sum_{j=1}^{n} \theta_j \wedge \overline{\theta_j}.$$

From this, we compute that

$$\omega^{\wedge n} = \omega \wedge \cdots \wedge \omega = n! \cdot \frac{i^n}{2^n} (\theta_1 \wedge \overline{\theta_1}) \wedge \cdots \wedge (\theta_n \wedge \overline{\theta_n}) = n! \cdot vol(g),$$

and so the volume form on the underlying oriented Riemannian manifold is given by Wirtinger's formula

$$vol(g) = \frac{1}{n!} \omega^{\wedge n}.$$

If we suppose in addition that M is compact, then we can conclude that

$$\mathrm{vol}(M) = \int_M vol(g) = \frac{1}{n!} \int_M \omega^{\wedge n}.$$

Since the volume of M is necessarily nonzero, it follows from Stokes' theorem that $\omega^{\wedge n}$ cannot be exact, and therefore that ω itself can never be an exact form.

Let $N \subseteq M$ be a complex submanifold, with the induced Hermitian metric. We then have $\omega_N = i^*\omega$, and if we set $m = \dim N$, then

$$\mathrm{vol}(N) = \frac{1}{m!} \int_N i^* \omega^{\wedge m}.$$

In particular, the volume of any submanifold is given by the integral of a globally defined differential form on M, which is very special to complex manifolds.

Example 10.1. The flat metric on $\mathbb{C}^n$ from Example 9.6 induces a Hermitian metric h_M on every complex torus $M = \mathbb{C}^n/\Lambda$. To compute the volume of M, we choose a fundamental domain $D \subseteq \mathbb{C}^n$ for the lattice; then the interior of D maps isomorphically to its image in M, and so

$$\mathrm{vol}(M) = \int_M vol(g_M) = \int_D vol(g) = \int_D d\mu$$

is the usual Lebesgue measure of D.

Introduction to cohomology. Sheaves are a useful tool for relating local to global data. We begin with a nice example, taken from "Principles of Algebraic Geometry" by Griffiths and Harris, that shows this passage from local to global.

Let M be a Riemann surface, not necessarily compact. Recall that a meromorphic function on M is a mapping $f\colon M \to \mathbb{C} \cup \{\infty\}$ that can locally be written as a quotient of two holomorphic functions, with denominator not identically zero. (Equivalently, a meromorphic function is a holomorphic mapping from M to the Riemann sphere $\mathbb{P}^1$, not identically equal to ∞.) In a neighborhood of any point $p \in M$, we can choose a holomorphic coordinate z centered at p, and write f in the form $\sum_{j \geq -N} a_j z^j$. The polar part of f is the sum $\pi_p(f) = \sum_{j<0} a_j z^j$; clearly f is holomorphic at p iff the polar part is zero.

A classical problem, named after Mittag-Leffler, is whether one can find a meromorphic function with prescribed polar parts at a discrete set of points $p_1, p_2, \dots$. One can approach this question from two different points of view.

For the first, let U_i be a small open neighborhood of p_i not containing any of the other points, and let π_i be the desired polar part at p_i. Also let $U_0 = M \setminus \{p_1, p_2, \dots\}$, and set $\pi_0 = 0$. On the intersection $U_i \cap U_j$, the difference $g_{i,j} = \pi_i - \pi_j$ is a holomorphic function. Now if we can find a meromorphic function f with those polar parts, then $f - \pi_i$ is holomorphic on U_i, and so $g_{i,j} = (f - \pi_j) - (f - \pi_i)$ is actually the difference of two holomorphic functions. Conversely, if there are holomorphic functions $f_i \in \mathscr{O}_M(U_i)$ such that $g_{i,j} = f_j - f_i$, then the individual functions $f_i + \pi_i$ agree on overlaps, and therefore define a global meromorphic function with the correct polar parts.

Note that $g_{i,j} + g_{j,k} = g_{i,k}$ on $U_i \cap U_j \cap U_k$. If we let $\mathbf{U}$ denote the given open cover, and $\mathbf{g}$ the collection of holomorphic functions $g_{i,j} \in \mathscr{O}_M(U_i \cap U_j)$, then we can summarize our observations as follows: Whether or not the Mittag-Leffler problem has a solution is measured by the class of $\mathbf{g}$ in the vector space

$$H^1(\mathbf{U}, \mathscr{O}_M) = Z^1(\mathbf{U}, \mathscr{O}_M)/B^1(\mathbf{U}, \mathscr{O}_M);$$

here $Z^1(\mathbf{U}, \mathscr{O}_M) = \left\{ \mathbf{g} \;\middle|\; g_{i,j} + g_{j,k} = g_{i,k} \text{ on } U_i \cap U_j \cap U_k \right\}$ is the space of so-called 1-cocycles, and $B^1(\mathbf{U}, \mathscr{O}_M) = \left\{ \mathbf{g} \;\middle|\; g_{i,j} = f_j - f_i \text{ for suitable } f_i \in \mathscr{O}_M(U_i) \right\}$ the space of 1-coboundaries. The quotient is the first *Čech cohomology group* for the sheaf $\mathscr{O}_M$ and the given open cover.

The second point of view is more analytic in nature. With the same notation as above, let ρ_i be a smooth function with compact support inside U_i, and identically equal to 1 in a neighorhood of the point p_i. Then

$$\omega = \sum_{i=0}^{\infty} \bar{\partial}(\rho_i \pi_i) = \sum_{i=0}^{\infty} \pi_i \cdot \bar{\partial}\rho_i$$

is a smooth $(0,1)$-form on M, identically equal to zero in a neighborhood of each point p_i. Suppose now that $\omega = \bar{\partial}\phi$ for some smooth function ϕ on M. Then ϕ is holomorphic in a neighborhood of each p_i, and the difference $f = \sum_i \rho_i \pi_i - \phi$ is holomorphic on U_0, and clearly has the correct polar part π_i at each point p_i. Since the converse is easily shown to be true as well, we arrive at the following conclusion: Whether or not the Mittag-Leffler problem has a solution is measured by the first Dolbeault cohomology group

$$H^{0,1}(M) = \frac{\left\{ \omega \in A^{0,1}(M) \;\middle|\; \bar{\partial}\omega = 0 \right\}}{\left\{ \bar{\partial}\phi \;\middle|\; \phi \in A^{0,0}(M) \right\}}.$$

Since we already know that $H^{0,1}(\mathbb{C}) = 0$ (by Proposition 8.5), we deduce the well-known fact that the Mittag-Leffler problem can always be solved on $\mathbb{C}$.

To summarize: Since the problem can always be solved locally, the only issue is the existence of a global solution. In either approach, the obstruction to finding a global solution lies in a certain cohomology group. In fact, as we will later see, $H^1(\mathbf{U}, \mathscr{O}_M) \simeq H^{0,1}(M)$.

Sheaves. We now introduce the useful concept of sheaves.

Definition 10.2. Let X be a topological space. A *sheaf* (of abelian groups) on X assigns to every open set $U \subseteq X$ a group $\mathscr{F}(U)$, called the *sections* of the sheaf, and to every inclusion $V \subseteq U$ a *restriction homomorphism* $\rho_V^U \colon \mathscr{F}(U) \to \mathscr{F}(V)$, subject to the following two conditions:

1. If $W \subseteq V \subseteq U$ are open sets, then $\rho_W^V \circ \rho_V^U = \rho_W^U$. One can therefore write $s|_V$ in place of $\rho_V^U(s)$ without loss of information.
2. If $s_i \in \mathscr{F}(U_i)$ is a collection of sections satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a unique $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$, where $U = \bigcup_{i \in I} U_i$.

In practice, a sheaf often has additional structure: for instance, we say that $\mathscr{F}$ is a sheaf of rings if every $\mathscr{F}(U)$ is a (commutative) ring, and if the restriction maps are ring homomorphisms. Similarly, there are sheaves of vector spaces, etc. For clarity, we sometimes denote the set of sections of $\mathscr{F}$ by the symbol $\Gamma(U, \mathscr{F})$ instead of the $\mathscr{F}(U)$ in the definition.

Example 10.3. A geometric structure $\mathscr{O}$ on a topological space X is a sheaf of rings: each $\mathscr{O}(U)$ is a subring of the ring of continuous functions on U, and the conditions in the two definitions are more or less identical.

Example 10.4. Let $\pi \colon E \to X$ be a vector bundle on X. Then assigning to every open set $U \subseteq X$ the space of continuous sections of the vector bundle over U defines a sheaf of vector spaces on X. When E is smooth (or holomorphic), we usually consider smooth (or holomorphic) sections instead.

On a complex manifold M, there are by and large three interesting classes of sheaves. The first are the so-called *locally constant sheaves*; for example, assigning to every open set U the set of locally constant maps from U into $\mathbb{Z}$ defines a sheaf $\mathbb{Z}_M$; one similarly defines $\mathbb{R}_M$ and $\mathbb{C}_M$. Such sheaves contain information about M as a topological space: for instance, $\Gamma(M, \mathbb{C}_M)$ is a $\mathbb{C}$-vector space whose dimension equals the number of connected components of M (since a locally constant functions has to be constant on each connected component).

The second class of sheaves are sections of smooth vector bundles, as in Example 10.4 above. The most important examples are the sheaf of sections of the tangent bundle, which assigns to every open set $U \subseteq M$ the space of smooth vector fields on U, and sheaves of differential forms. We let $\mathscr{A}^k$ be the sheaf that assigns to an open set U the space of smooth k-forms on U (these are sections of the vector bundle $\bigwedge^k T_{\mathbb{R}}^* M$). Likewise, the sections of the sheaf $\mathscr{A}^{p,q}$ are the (p,q)-forms on U. Such sheaves contain information about M as a smooth manifold, and are very useful for doing calculus.

The third class are those sheaves that are connected to the complex structure on M. Examples are the structure sheaf $\mathscr{O}_M$, whose sections are the holomorphic functions; the sheaves Ω^p_M, where $\Omega^p_M(U)$ is the space of holomorphic forms of type $(p,0)$ on U; the sheaf of units $\mathscr{O}^*_M$, defined by letting $\mathscr{O}^*_M(U)$ be the set of nowhere vanishing holomorphic functions on U; and, more generally, the sheaf of holomorphic sections of any holomorphic vector bundle on M.

Stalks and operations. Let $\mathscr{F}$ be a sheaf on a topological space X, and let $x \in X$ be a point. The *stalk* of the sheaf is the direct limit

$$\mathscr{F}_x = \lim_{U \ni x} \mathscr{F}(U),$$

taken over all open neighborhoods of the point. The stalk is again an abelian group; it is a ring (or vector space) if $\mathscr{F}$ is a sheaf of rings (or vector spaces). We think of elements of the stalk as germs of sections at x.

Example 10.5. On a complex manifold M, the local ring $\mathscr{O}_{M,p}$ is the stalk of the sheaf $\mathscr{O}_M$ at the point p.

A *morphism* of sheaves $f \colon \mathscr{F} \to \mathscr{G}$ is a collection of group homomorphisms $f_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$, compatible with restriction maps in the sense that $\rho^U_V \circ f_U = f_V \circ \rho^U_V$ for every inclusion $V \subseteq U$. If each f_U is the inclusion of a subgroup, we say that $\mathscr{F}$ is a *subsheaf* of $\mathscr{G}$.

The *kernel* of a morphism of sheaves is the subsheaf of $\mathscr{F}$ defined by setting

$$\Gamma(U, \ker f) = \left\{\, s \in \mathscr{F}(U) \;\middle|\; f_U(s) = 0 \,\right\};$$

it is not hard to verify that $\ker f$ is indeed a sheaf. A morphism of sheaves also has an *image* $\operatorname{im} f$, which is a subsheaf of $\mathscr{G}$; but the definition is more complicated since the groups $\operatorname{im} f_U$ do not form a sheaf. To ensure that the second condition in Definition 10.2 is satisfied, we are forced instead to set

$$\Gamma(U, \operatorname{im} f) = \left\{\, s \in \mathscr{G}(U) \;\middle|\; s|_{U_i} \in \operatorname{im} f_{U_i} \text{ for some open cover } U = \bigcup_{i \in I} U_i \,\right\}.$$

We say that f is *injective* if $\ker f = 0$, and that f is *surjective* if $\operatorname{im} f = \mathscr{G}$. Finally, we say that a sequence of sheaves

$$\mathscr{F}^0 \xrightarrow{f^0} \mathscr{F}^1 \xrightarrow{f^1} \mathscr{F}^2 \longrightarrow \cdots \longrightarrow \mathscr{F}^{k-1} \xrightarrow{f^{k-1}} \mathscr{F}^k$$

is a *complex* if $f_{i+1} \circ f_i = 0$ for every i, and that it is *exact* if $\ker f_{i+1} = \operatorname{im} f_i$ at all places.

Example 10.6. If $\mathscr{F}$ is a subsheaf of $\mathscr{G}$, one can also define a quotient sheaf $\mathscr{G}/\mathscr{F}$, in such a way that there is an exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{G}/\mathscr{F} \to 0$. It is a good exercise to work out the correct definition.

The following example illustrates these notions; it is one of the most important exact sequences of sheaves on a complex manifold M.

Example 10.7. On a complex manifold M, the so-called *exponential sequence*

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow \mathscr{O}_M \xrightarrow{\exp} \mathscr{O}^*_M \longrightarrow 0$$

is an exact sequence of sheaves. (The group operation on $\mathscr{O}^*_M(U)$ is multiplication.) The first map is given by the inclusion $\mathbb{Z}_M(U) \subseteq \mathscr{O}_M(U)$, using that locally constant functions are holomorphic. The second map takes a holomorphic function $f \in \mathscr{O}_M(U)$ to the nowhere vanishing holomorphic function $\exp_U(f) = e^{2\pi i f}$. It is easy to see that the sequence is exact at $\mathbb{Z}_M$ and at $\mathscr{O}_M$; in fact, if $e^{2\pi i f} = 1$ for some holomorphic function f, then f is integer-valued, and hence locally constant.

Exactness at $\mathscr{O}_M^*$ means the surjectivity of exp; according to the definition above, this amounts to saying that a nowhere vanishing holomorphic function g can *locally* be written in the form $e^{2\pi i f}$. After choosing local coordinates, we can reduce to the case $g \in \mathscr{O}(D)$, where $D \subseteq \mathbb{C}^n$ is a small polydisk. After choosing a suitable branch of the logarithm, we can then take $f = \log g$ on D.

Note that the individual maps $\exp_U \colon \mathscr{O}_M(U) \to \mathscr{O}_M^*(U)$ need not be surjective; with $M = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\}$, for example, the holomorphic function z cannot be written in the form $e^{2\pi i f}$ with f holomorphic on U.

The example shows that a morphism $f \colon \mathscr{F} \to \mathscr{G}$ can be surjective, even though the individual maps $f_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$ are not.

We note that a morphism $f \colon \mathscr{F} \to \mathscr{G}$ always induces homomorphisms $f_x \colon \mathscr{F}_x \to \mathscr{G}_x$ between stalks. The following proposition shows that injectivity, surjectivity, and so forth, can be verified at the level of stalks; this means that they are local properties.

Proposition 10.8. *Let $f \colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then f is surjective (resp., injective) iff for every point $x \in X$, the induced map on stalks $f_x \colon \mathscr{F}_x \to \mathscr{G}_x$ is injective (resp., surjective). Likewise, a sequence of sheaves*

$$\mathscr{F}^0 \xrightarrow{f^0} \mathscr{F}^1 \xrightarrow{f^1} \mathscr{F}^2 \longrightarrow \cdots \longrightarrow \mathscr{F}^{k-1} \xrightarrow{f^{k-1}} \mathscr{F}^k$$

is exact iff the induced sequence of abelian groups

$$\mathscr{F}_x^0 \xrightarrow{f_x^0} \mathscr{F}_x^1 \xrightarrow{f_x^1} \mathscr{F}_x^2 \longrightarrow \cdots \longrightarrow \mathscr{F}_x^{k-1} \xrightarrow{f_x^{k-1}} \mathscr{F}_x^k$$

is exact for every point $x \in X$.

Sheaf cohomology. The following lemma is easy to prove from the definitions.

Lemma 10.9. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves on a topological space X, then $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X)$ is an exact sequence of abelian groups.*

In general, the map $\mathscr{F}(X) \to \mathscr{F}''(X)$ need not be surjective; we have already seen an example of this above. But in practice, one often needs to know whether or not a given section of $\mathscr{F}''$ can be lifted to a section of $\mathscr{F}$. Sheaf cohomology solves this problem by giving us a long exact sequence of abelian groups

$$\begin{array}{l} 0 \longrightarrow H^0(X,\mathscr{F}') \longrightarrow H^0(X,\mathscr{F}) \longrightarrow H^0(X,\mathscr{F}'') \longrightarrow \\ \longrightarrow H^1(X,\mathscr{F}') \longrightarrow H^1(X,\mathscr{F}) \longrightarrow H^1(X,\mathscr{F}'') \longrightarrow \\ \longrightarrow H^2(X,\mathscr{F}') \longrightarrow H^2(X,\mathscr{F}) \longrightarrow H^2(X,\mathscr{F}'') \longrightarrow \cdots \end{array}$$

Here $H^0(X, \mathscr{F}) = \mathscr{F}(X)$, and so the higher cohomology groups $H^i(X, \mathscr{F})$ extend the sequence in Lemma 10.9. This means that a section in $\mathscr{F}''(X)$ can be lifted to a section in $\mathscr{F}(X)$ iff its image in the first cohomology group $H^1(X, \mathscr{F}')$ is zero.

To define the cohomology groups of a sheaf, we introduce the following notion: A sheaf $\mathscr{F}$ on a topological space is called *flabby* if the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$

is surjective for every open set $U \subseteq X$. With flabby sheaves, taking global sections preserves exactness.

Lemma 10.10. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves on a topological space X, and if $\mathscr{F}'$ is flabby, then*

$$0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0$$

is an exact sequence of abelian groups.

Proof. Let $\alpha\colon \mathscr{F}' \to \mathscr{F}$ and $\beta\colon \mathscr{F} \to \mathscr{F}''$ denote the maps. By virtue of Lemma 10.9, it suffices to show that $\beta_X\colon \mathscr{F}(X) \to \mathscr{F}''(X)$ is surjective. This is most easily done by using Zorn's lemma. Fix a global section $s'' \in \mathscr{F}''(X)$, and consider the set of all pairs (U, s), where $U \subseteq X$ is open and $s \in \mathscr{F}(U)$ satisfies $\beta_U(s) = s''|_U$. It is clear that this set is nonempty, because β is surjective on stalks.

We put a partial order on our set of pairs by declaring that $(U_1, s_1) \leq (U_2, s_2)$ if $U_1 \subseteq U_2$ and $s_2|_{U_1} = s_1$. Since $\mathscr{F}$ is a sheaf, every chain $\{(U_i, s_i)\}_{i\in I}$ has an upper bound (U, s): take $U = \bigcup_{i\in I} U_i$ and let $s \in \mathscr{F}(U)$ be the unique section with $s|_{U_i} = s_i$ for all $i \in I$. By Zorn's lemma, there is a maximal element (U_{max}, s_{max}). To complete the proof, we need to show that $U_{max} = X$.

To that end, let $x \in X$ be any point. Then $\beta_x\colon \mathscr{F}_x \to \mathscr{F}''_x$ is onto, and so we can find a pair (U, s) with $x \in U$. On $V = U \cap U_{max}$, we now have two sections lifting s'', and so by Lemma 10.9, there is a unique section $s' \in \mathscr{F}'(U \cap U_{max})$ with $\alpha_V(s') = (s_{max} - s)|_V$. But now $\mathscr{F}'$ is flabby, and so we can find $t' \in \mathscr{F}'(U)$ with $t'|_V = s'$; then $s_{max} \in \mathscr{F}(U_{max})$ and $s + \alpha_U(t') \in \mathscr{F}(U)$ agree on V, and therefore define a section in $\mathscr{F}(U \cup U_{max})$ that still maps to s''. By maximality, we have $U \cup U_{max} = U_{max}$, and therefore $x \in U_{max}$. This proves that $U_{max} = X$, and shows that $s_{max} \in \mathscr{F}(X)$ satisfies $\beta_X(s_{max}) = s''$. $\square$

Next, we show that any sheaf has a canonical resolution by flabby sheaves. Given any sheaf $\mathscr{F}$, let $T(\mathscr{F}) = \bigsqcup_{x\in X} \mathscr{F}_x$ be the disjoint union of its stalks; we can then define the *sheaf of discontinuous sections* $\operatorname{ds}\mathscr{F}$ by setting

$$\Gamma(U, \operatorname{ds}\mathscr{F}) = \left\{\, s\colon U \to T(\mathscr{F}) \;\middle|\; s(x) \in \mathscr{F}_x \text{ for all } x \in X \,\right\}.$$

It is obvious from the definition that $\operatorname{ds}\mathscr{F}$ is a flabby sheaf; moreover, we have an injective map of sheaves $\varepsilon\colon \mathscr{F} \to \operatorname{ds}\mathscr{F}$, taking a section $s \in \mathscr{F}(U)$ to the map $x \mapsto s_x$. This construction gives us an exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\ \varepsilon\ } \mathscr{F}^0 \xrightarrow{\ d^0\ } \mathscr{F}^1 \xrightarrow{\ d^1\ } \mathscr{F}^2 \xrightarrow{\ d^2\ } \cdots, \tag{10.11}$$

in which the $\mathscr{F}^i$ are flabby sheaves, as follows: Define $\mathscr{F}^0 = \operatorname{ds}\mathscr{F}$, and let $\varepsilon\colon \mathscr{F} \to \mathscr{F}^0$ be the map from above. Next, form the quotient sheaf $\mathscr{G}^0 = \mathscr{F}^0/\operatorname{im}\varepsilon$, let $\mathscr{F}^1 = \operatorname{ds}\mathscr{G}^0$, and let $d^1\colon \mathscr{F}^0 \to \mathscr{F}^1$ be the composition of the two natural maps. Continuing in this way, we obtain a commutative diagram of the type

$$\begin{tikzcd}
\mathscr{F} \arrow[dr, "\varepsilon"] & & & & \mathscr{G}^1 \arrow[dr] & & & & \mathscr{G}^3 \arrow[dr] & \\
& \mathscr{F}^0 \arrow[rr, "d^0"] \arrow[dr] & & \mathscr{F}^1 \arrow[rr, "d^1"] \arrow[ur] & & \mathscr{F}^2 \arrow[rr, "d^2"] \arrow[dr] & & \mathscr{F}^3 \arrow[rr] \arrow[ur] & & \cdots \\
& & \mathscr{G}^0 \arrow[ur] & & & & \mathscr{G}^2 \arrow[ur] & & &
\end{tikzcd} \tag{10.12}$$

continuing to the right; at each stage, $\mathscr{F}^k = \operatorname{ds}\mathscr{G}^{k-1}$, and $\mathscr{G}^k$ is the quotient of $\mathscr{F}^k$ by its subsheaf $\mathscr{G}^{k-1}$. Since the diagonal sequences are all exact, it is not hard to prove (by looking at stalks) that (10.11) is itself exact. We refer to it as the *Godement resolution* of the sheaf $\mathscr{F}$.

Definition 10.13. For a sheaf $\mathscr{F}$ on a topological space X, we define $H^i(X, \mathscr{F})$ to be the i-th cohomology group of the complex of abelian groups

$$0 \longrightarrow \mathscr{F}^0(X) \longrightarrow \mathscr{F}^1(X) \longrightarrow \mathscr{F}^2(X) \longrightarrow \cdots.$$

It follows from Lemma 10.9 that the sequence $0 \to \mathscr{F}(X) \to \mathscr{F}^0(X) \to \mathscr{F}^1(X)$ is exact, and therefore that $H^0(X, \mathscr{F}) \simeq \mathscr{F}(X)$. Note also that when $\mathscr{F}$ is a sheaf of vector spaces, each $H^i(X, \mathscr{F})$ is again a vector space. As promised, we always have a long exact sequence in cohomology.

Proposition 10.14. *A short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of sheaves gives rise to a long exact sequence of cohomology groups.*

Proof. A morphism $f \colon \mathscr{F} \to \mathscr{G}$ induces maps on stalks, and hence a morphism $\operatorname{ds}\mathscr{F} \to \operatorname{ds}\mathscr{G}$ between the sheaves of discontinuous sections. Using this fact, one can easily show that the Godement resolutions for the three sheaves fit into a commutative diagram

$$\begin{tikzcd}
\mathscr{F}'^0 \arrow[r] \arrow[d] & \mathscr{F}'^1 \arrow[r] \arrow[d] & \mathscr{F}'^2 \arrow[r] \arrow[d] & \cdots \\
\mathscr{F}^0 \arrow[r] \arrow[d] & \mathscr{F}^1 \arrow[r] \arrow[d] & \mathscr{F}^2 \arrow[r] \arrow[d] & \cdots \\
\mathscr{F}''^0 \arrow[r] & \mathscr{F}''^1 \arrow[r] & \mathscr{F}''^2 \arrow[r] & \cdots
\end{tikzcd}$$

with exact columns. Since each $\mathscr{F}'^k$ is flabby, it follows from Lemma 10.10 that, even after taking global sections, the columns in

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'^0(X) \arrow[r] \arrow[d] & \mathscr{F}'^1(X) \arrow[r] \arrow[d] & \mathscr{F}'^2(X) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathscr{F}^0(X) \arrow[r] \arrow[d] & \mathscr{F}^1(X) \arrow[r] \arrow[d] & \mathscr{F}^2(X) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathscr{F}''^0(X) \arrow[r] & \mathscr{F}''^1(X) \arrow[r] & \mathscr{F}''^2(X) \arrow[r] & \cdots
\end{tikzcd}$$

are short exact sequences of abelian groups. The long exact sequence of cohomology groups is then obtained by applying a form of the Snake Lemma, which is a basic result in homological algebra. $\square$

To conclude our discussion of flabby sheaves, we would like to show that the higher cohomology groups of flabby sheaves are zero. We begin with a small lemma.

Lemma 10.15. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence with $\mathscr{F}'$ and $\mathscr{F}$ flabby, then $\mathscr{F}''$ is also flabby.*

Proof. For any open subset $U \subseteq X$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{F}(X) & \longrightarrow & \mathscr{F}''(X) \\ \downarrow & & \downarrow \\ \mathscr{F}(U) & \longrightarrow & \mathscr{F}''(U). \end{array}$$

The surjectivity of the two horizontal maps is due to Lemma 10.10, and that of the vertical restriction map comes from the flabbiness of $\mathscr{F}$. We conclude that $\mathscr{F}''(X) \to \mathscr{F}''(U)$ is also surjective, proving that $\mathscr{F}''$ is flabby. ☐

We can now prove that flabby sheaves have trivial cohomology.

Proposition 10.16. *If $\mathscr{F}$ is a flabby sheaf, then $H^i(X, \mathscr{F}) = 0$ for $i > 0$.*

Proof. According to the preceding lemma, the quotient of a flabby sheaf by a flabby subsheaf is again flabby. This fact implies that in (10.12), all the the sheaves $\mathscr{G}^j$ are also flabby sheaves. Consequently, the entire diagram remains exact after taking global sections, which shows that $0 \to \mathscr{F}(X) \to \mathscr{F}^0(X) \to \mathscr{F}^1(X) \to \cdots$ is an exact sequence of abelian groups. But this means that $H^i(X, \mathscr{F}) = 0$ for $i > 0$. ☐