Math 535
Solutions to Midterm 2

Tuesday, April 9, 2024

1. Let A be an $n \times n$-matrix over a field k, with minimal polynomial $m_A(x)$. Let $f(x) \in k[x]$ be an arbitrary polynomial. Show that the matrix $f(A)$ is invertible if and only if $f(x)$ and $m_A(x)$ are relatively prime in the polynomial ring $k[x]$.

   Solution: Suppose first that $f(x)$ and $m_A(x)$ are relatively prime. Since $k[x]$ is a PID, there are polynomials $p(x), q(x) \in k[x]$ such that $1 = p(x)f(x) + q(x)m_A(x)$. If we evaluate this identity on the matrix A, we get

   $$I_n = p(A)f(A) + q(A)m_A(A) = p(A)f(A),$$

   and so $f(A)$ is an invertible matrix with inverse $p(A)$. Conversely, suppose that the matrix $f(A)$ is invertible. In order to show that $m_A(x)$ and $f(x)$ are relatively prime, it is enough to prove that they have no common roots (say in the algebraic closure $\bar{k}$ of the field k). The roots of the minimal polynomial are exactly the eigenvalues of A, and so we need to argue that no eigenvalue of the matrix A can be a root of $f(x)$. Let $\lambda \in \bar{k}$ be an eigenvalue, and let $v \in \bar{k}^n$ be a nonzero vector for which $Av = \lambda v$. Then we get

   $$f(\lambda)v = f(A)v \neq 0,$$

   because $f(A)$ is invertible. It follows that $f(\lambda) \neq 0$.

2. How many similarity classes are there of real 8×8-matrices with minimal polynomial $m_A(x) = (x-1)^3(x+1)^2$ and characteristic polynomial $f_A(x) = (x-1)^5(x+1)^3$? List a representative for each class.

   Solution: The roots of the characteristic polynomial, and hence the eigenvalues of the matrix, are the two real numbers ± 1. We know that every real matrix with real eigenvalues is similar to a real matrix in Jordan canonical form, unique up to the ordering of the Jordan blocks. From the minimal polynomial, we see that all Jordan blocks for the eigenvalue $+1$ have size ≤ 3, and that there is at least one Jordan block of size 3; likewise, all Jordan blocks for the eigenvalue -1 have size ≤ 2, and there is at least one Jordan block of size 2. From the characteristic polynomial, we see that the sizes of all the Jordan blocks for the eigenvalue $+1$ must add up to 5. Denote by $J_k(\lambda)$ a Jordan block of size k and eigenvalue λ. Then there are 2 possibilities:

(a) $J_3(1) \oplus J_2(1) \oplus J_2(-1) \oplus J_1(-1)$

(b) $J_3(1) \oplus J_1(1) \oplus J_1(1) \oplus J_2(-1) \oplus J_1(-1)$

3. Let k be a field of characteristic $\neq 2$, and let $f(x) \in k[x]$ be a cubic polynomial. Suppose that the discriminant of $f(x)$ is a square in k. Prove that $f(x)$ is either irreducible or a product of three linear polynomials.

   Solution: Here is a Galois theory argument that works if $f(x)$ is separable. Let E be the splitting field of the polynomial $f(x)$, and let $G = \text{Gal}(E/k)$ be the Galois group. Because a cubic polynomial has three roots, G is isomorphic to a subgroup of the symmetric group S_3. In the case at hand, the discriminant is a square, and so we know from class that G is a subgroup of the alternating group A_3, which is a cyclic group of order 3. Now there are two possibilities:

   (a) If G is the trivial group, we get $(E : k) = |G| = 1$, and so $f(x)$ splits over k as a product of three linear polynomials

   (b) If G is a cyclic group of order 3, then G is generated by a 3-cycle, and therefore acts transitively on the roots of $f(x)$. This implies that $f(x)$ cannot factor in $k[x]$, and so it must be irreducible.

   Without assuming that $f(x)$ is separable, we can argue as follows. Suppose that $f(x)$ is reducible; being a cubic, it must then have a root $\alpha \in k$. This gives us a factorization $f(x) = (x - \alpha)g(x)$ with $g(x) \in k[x]$ a quadratic polynomial. We need to argue that $g(x)$ is the product of two linear factors. Let E be the splitting field of $g(x)$ and let $\beta, \gamma \in E$ be the two roots. Arguing by contradiction, let's assume that $\beta, \gamma \notin k$. The discriminant of $f(x)$ is

$$(\alpha - \beta)^2(\alpha - \gamma)^2(\beta - \gamma)^2,$$

   and since this is the square of an element in k, it follows that

$$(\alpha - \beta)(\alpha - \gamma)(\beta - \gamma) \in k.$$

   But $\beta + \gamma$ and $\beta\gamma$ are elements of k, and so

$$(\alpha - \beta)(\alpha - \gamma) = \alpha^2 - (\beta + \gamma)\alpha + \beta\gamma$$

   is a nonzero element of k. We conclude that $\beta - \gamma \in k$, and therefore that 2β and 2γ are in k. As $\text{char}(k) \neq 2$, this implies that $\beta, \gamma \in k$, which is a contradiction.

4. Determine the signature of the symmetric bilinear form on $\mathbb{R}^3$ represented by the matrix

$$B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

Solution: Denote by (x, y, z) the coordinates on $\mathbb{R}^3$. The associated quadratic form is

$$B(x, y, z) = (x, y, z) \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2xy - 2xz + y^2 - z^2.$$

After completing the square, we get

$$B(x, y, z) = (x + y)^2 - (x + z)^2,$$

and this visibly has signature $(1, 1)$: there is one positive coefficient and one negative coefficient.

Alternatively, we can compute the characteristic polynomial $f_B(x) = x(x^2 - 3)$, and note that the three eigenvalues of the matrix B are therefore 0 and $\pm\sqrt{3}$. By a homework problem, there is a basis of eigenvectors in which the bilinear form B is represented by the matrix

$$\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the signature is again $(1, 1)$.